

COMPLEMENTS ON DISCONNECTED REDUCTIVE GROUPS

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Dedicated to the memory of Robert Steinberg

ABSTRACT. We present various results on disconnected reductive groups, in particular about the characteristic 0 representation theory of such groups over finite fields.

1. INTRODUCTION

Let \mathbf{G} be a (possibly disconnected) linear algebraic group over an algebraically closed field. We assume that the connected component \mathbf{G}^0 is reductive, and then call \mathbf{G} a (possibly disconnected) reductive group. This situation was studied by Steinberg in [St] where he introduced the notion of quasi-semi-simple elements.

Assume now that \mathbf{G} is over an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field \mathbb{F}_q , defined over \mathbb{F}_q with corresponding Frobenius endomorphism F . Let \mathbf{G}^1 be an F -stable connected component of \mathbf{G} . We want to study $(\mathbf{G}^0)^F$ -class functions on $(\mathbf{G}^1)^F$; if \mathbf{G}^1 generates \mathbf{G} , they coincide with \mathbf{G}^F -class functions on $(\mathbf{G}^1)^F$.

This setting we adopt here is also taken up by Lusztig in his series of papers on disconnected groups [Lu] and is slightly more general than the setting of [DM94], where we assumed that \mathbf{G}^1 contains an F -stable quasi-central element. A detailed comparison of both situations is done in the next section.

As the title says, this paper is a series of complements to our original paper [DM94] which are mostly straightforward developments that various people asked us about and that, as far as we know, have not appeared in the literature; we thank in particular Olivier Brunat, Gerhard Hiss, Cheryl Praeger and Karine Sorlin for asking these questions.

In section 2 we show how quite a few results of [DM94] are still valid in our more general setting.

In section 3 we take a “global” viewpoint to give a formula for the scalar product of two Deligne-Lusztig characters on the whole of \mathbf{G}^F .

In section 4 we show how to extend to disconnected groups the formula of Steinberg [St, 15.1] counting unipotent elements.

In section 5 we extend the theorem that tensoring Lusztig induction with the Steinberg character gives ordinary induction.

In section 6 we give a formula for the characteristic function of a quasi-semi-simple class, extending the case of a quasi-central class which was treated in [DM94].

In section 7 we show how to classify quasi-semi-simple conjugacy classes, first for a (possibly disconnected) reductive group over an arbitrary algebraically closed field, and then over \mathbb{F}_q .

Finally, in section 8 we extend to our setting previous results on Shintani descent.

2. PRELIMINARIES

In this paper, we consider a (possibly disconnected) algebraic group \mathbf{G} over $\overline{\mathbb{F}}_q$ (excepted at the beginning of section 7 where we accept an arbitrary algebraically closed field), defined over \mathbb{F}_q with corresponding Frobenius automorphism F . If \mathbf{G}^1 is an F -stable component of \mathbf{G} , we call class functions on $(\mathbf{G}^1)^F$ the complex-valued functions invariant under $(\mathbf{G}^0)^F$ -conjugacy (or equivalently under $(\mathbf{G}^1)^F$ -conjugacy). Note that if \mathbf{G}^1 does not generate \mathbf{G} , there may be less functions invariant by \mathbf{G}^F -conjugacy than by $(\mathbf{G}^1)^F$ -conjugacy; but the propositions we prove will apply *in particular* to the \mathbf{G}^F -invariant functions so we do not lose any generality. The class functions on $(\mathbf{G}^1)^F$ are provided with the scalar product $\langle f, g \rangle_{(\mathbf{G}^1)^F} = |(\mathbf{G}^1)^F|^{-1} \sum_{h \in (\mathbf{G}^1)^F} f(h) \overline{g(h)}$. We call \mathbf{G} reductive when \mathbf{G}^0 is reductive.

When \mathbf{G} is reductive, following [St] we call quasi-semi-simple an element which normalizes a pair $\mathbf{T}^0 \subset \mathbf{B}^0$ of a maximal torus of \mathbf{G}^0 and a Borel subgroup of \mathbf{G}^0 . Following [DM94, 1.15], we call quasi-central a quasi-semi-simple element σ which has maximal dimension of centralizer $C_{\mathbf{G}^0}(\sigma)$ (that we will also denote by $\mathbf{G}^{0\sigma}$) amongst all quasi-semi-simple elements of $\mathbf{G}^0 \cdot \sigma$.

In the sequel, we fix a reductive group \mathbf{G} and (excepted in the next section where we take a “global” viewpoint) an F -stable connected component \mathbf{G}^1 of \mathbf{G} . In most of [DM94] we assumed that $(\mathbf{G}^1)^F$ contained a quasi-central element. Here we do not assume this. Note however that by [DM94, 1.34] \mathbf{G}^1 contains an element σ which induces an F -stable quasi-central automorphism of \mathbf{G}^0 . Such an element will be enough for our purpose, and we fix one from now on.

By [DM94, 1.35] when $H^1(F, Z\mathbf{G}^0) = 1$ then $(\mathbf{G}^1)^F$ contains quasi-central elements. Here is an example where $(\mathbf{G}^1)^F$ does not contain quasi-central elements.

Example 2.1. Take $s = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ where $\xi \in \mathbb{F}_q - \mathbb{F}_q^2$, take $\mathbf{G}^0 = \mathrm{SL}_2$ and let $\mathbf{G} = \langle \mathbf{G}^0, s \rangle \subset \mathrm{GL}_2$ endowed with the standard Frobenius endomorphism on GL_2 , so that s is F -stable and $\mathbf{G}^F = \mathrm{GL}_2(\mathbb{F}_q)$. We take $\mathbf{G}^1 = \mathbf{G}^0 \cdot s$. Here quasi-central elements are central and coincide with $\mathbf{G}^0 \cdot s \cap Z\mathbf{G}$ which is nonempty since if $\eta \in \mathbb{F}_{q^2}$ is a square root of ξ then $\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \in \mathbf{G}^0 \cdot s \cap Z\mathbf{G}$; but $\mathbf{G}^0 \cdot s$ does not meet $(Z\mathbf{G})^F$. \square

In the above example $\mathbf{G}^1/\mathbf{G}^0$ is semisimple. No such example exists when $\mathbf{G}^1/\mathbf{G}^0$ is unipotent:

Lemma 2.2. *Let \mathbf{G}^1 be an F -stable connected component of \mathbf{G} such that $\mathbf{G}^1/\mathbf{G}^0$ is unipotent. Then $(\mathbf{G}^1)^F$ contains unipotent quasi-central elements.*

Proof. Let $\mathbf{T}^0 \subset \mathbf{B}^0$ be a pair of an F -stable maximal torus of \mathbf{G}^0 and an F -stable Borel subgroup of \mathbf{G}^0 . Then $N_{\mathbf{G}^F}(\mathbf{T}^0 \subset \mathbf{B}^0)$ meets $(\mathbf{G}^1)^F$, since any two F -stable pairs $\mathbf{T}^0 \subset \mathbf{B}^0$ are $(\mathbf{G}^0)^F$ -conjugate. Let su be the Jordan decomposition of an element of $N_{(\mathbf{G}^1)^F}(\mathbf{T}^0 \subset \mathbf{B}^0)$. Then $s \in \mathbf{G}^0$ since $\mathbf{G}^1/\mathbf{G}^0$ is unipotent, and u is F -stable, unipotent and still in $N_{(\mathbf{G}^1)^F}(\mathbf{T}^0 \subset \mathbf{B}^0)$ thus quasi-semi-simple, so is quasi-central by [DM94, 1.33]. \square

Note, however, that there may exist a unipotent quasi-central element σ which is rational as an automorphism but such that there is no rational element inducing the same automorphism.

Example 2.3. We give an example in $\mathbf{G} = \mathrm{SL}_5 \rtimes \langle \sigma' \rangle$ where $\mathbf{G}^0 = \mathrm{SL}_5$ has the standard rational structure over a finite field \mathbb{F}_q of characteristic 2 with $q \equiv 1 \pmod{5}$ and σ' is the automorphism of \mathbf{G}^0 given by $g \mapsto J^t g^{-1} J$ where J is the antidiagonal matrix with all non-zero entries equal to 1, so that σ' stabilizes the pair $\mathbf{T}^0 \subset \mathbf{B}^0$ where \mathbf{T}^0 is the maximal torus of diagonal matrices and \mathbf{B}^0 the Borel subgroup of upper triangular matrices, hence σ' is quasi-semi-simple. Let t be the diagonal matrix with entries (a, a, a^{-4}, a, a) where a^{q-1} is a non trivial 5-th root of unity $\zeta \in \mathbb{F}_q$. We claim that $\sigma = t\sigma'$ is as announced: it is still quasi-semi-simple; we have $\sigma^2 = t\sigma'(t) = tt^{-1} = 1$ so that σ is unipotent; we have ${}^F\sigma = {}^Ftt^{-1}\sigma = \zeta\sigma$, so that σ is rational as an automorphism but not rational. Moreover a rational element inducing the same automorphism must be of the form $z\sigma$ with z central in \mathbf{G}^0 and $z \cdot {}^Fz^{-1} = \zeta \mathrm{Id}$; but the center $Z\mathbf{G}^0$ is generated by $\zeta \mathrm{Id}$ and for any $z = \zeta^k \mathrm{Id} \in Z\mathbf{G}^0$ we have $z \cdot {}^Fz^{-1} = \zeta^{k(q-1)} \mathrm{Id} = \mathrm{Id} \neq \zeta \mathrm{Id}$. \square

As in [DM94] we call “Levi” of \mathbf{G} a subgroup \mathbf{L} of the form $N_{\mathbf{G}}(\mathbf{L}^0 \subset \mathbf{P}^0)$ where \mathbf{L}^0 is a Levi subgroup of the parabolic subgroup \mathbf{P}^0 of \mathbf{G}^0 . A particular case is a “torus” $N_{\mathbf{G}}(\mathbf{T}^0, \mathbf{B}^0)$ where $\mathbf{T}^0 \subset \mathbf{B}^0$ is a pair of a maximal torus of \mathbf{G}^0 and a Borel subgroup of \mathbf{G}^0 ; note that a “torus” meets all connected components of \mathbf{G} , while (contrary to what is stated erroneously after [DM94, 1.4]) this may not be the case for a “Levi”.

We call “Levi” of \mathbf{G}^1 a set of the form $\mathbf{L}^1 = \mathbf{L} \cap \mathbf{G}^1$ where \mathbf{L} is a “Levi” of \mathbf{G} and the intersection is nonempty; note that if \mathbf{G}^1 does not generate \mathbf{G} , there may exist several “Levis” of \mathbf{G} which have same intersection with \mathbf{G}^1 . Nevertheless \mathbf{L}^1 determines \mathbf{L}^0 as the identity component of $\langle \mathbf{L}^1 \rangle$.

We assume now that \mathbf{L}^1 is an F -stable “Levi” of \mathbf{G}^1 of the form $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$. If \mathbf{U} is the unipotent radical of \mathbf{P}^0 , we define $\mathbf{Y}_{\mathbf{U}}^0 = \{x \in \mathbf{G}^0 \mid x^{-1} \cdot {}^F x \in \mathbf{U}\}$ on which $(g, l) \in \mathbf{G}^F \times \mathbf{L}^F$ such that $gl \in \mathbf{G}^0$ acts by $x \mapsto gxl$, where $\mathbf{L} = N_{\mathbf{G}}(\mathbf{L}^0, \mathbf{P}^0)$. Along the same lines as [DM94, 2.10] we define

Definition 2.4. Let \mathbf{L}^1 be an F -stable “Levi” of \mathbf{G}^1 of the form $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ and let \mathbf{U} be the unipotent radical of \mathbf{P}^0 . For λ a class function on $(\mathbf{L}^1)^F$ and $g \in (\mathbf{G}^1)^F$ we set

$$R_{\mathbf{L}^1}^{\mathbf{G}^1}(\lambda)(g) = |(\mathbf{L}^1)^F|^{-1} \sum_{l \in (\mathbf{L}^1)^F} \lambda(l) \mathrm{Trace}((g, l^{-1}) \mid H_c^*(\mathbf{Y}_{\mathbf{U}}^0))$$

and for γ a class function on $(\mathbf{G}^1)^F$ and $l \in (\mathbf{L}^1)^F$ we set

$${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma)(l) = |(\mathbf{G}^1)^F|^{-1} \sum_{g \in (\mathbf{G}^1)^F} \gamma(g) \mathrm{Trace}((g^{-1}, l) \mid H_c^*(\mathbf{Y}_{\mathbf{U}}^0)).$$

In the above H_c^* denotes the ℓ -adic cohomology with compact support, where we have chosen once and for all a prime number $\ell \neq p$. In order to consider the virtual character $\mathrm{Trace}(x \mid H_c^*(\mathbf{X})) = \sum_i (-1)^i \mathrm{Trace}(x \mid H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell))$ as a complex character we chose once and for all an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

Writing $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ is an abuse of notation: the definition needs the choice of a \mathbf{P}^0 such that $\mathbf{L}^1 = N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$. Our subsequent statements will use an implicit choice. Under certain assumptions we will prove a Mackey formula (Theorem 2.6) which when true implies that $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ are independent of the choice of \mathbf{P}^0 .

By the same arguments as for [DM94, 2.10] (using that $(\mathbf{L}^1)^F$ is nonempty and [DM94, 2.3]) definition 2.4 agrees with the restriction to $(\mathbf{G}^1)^F$ and $(\mathbf{L}^1)^F$ of [DM94, 2.2].

The two maps $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and $*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ are adjoint with respect to the scalar products on $(\mathbf{G}^1)^F$ and $(\mathbf{L}^1)^F$.

We note the following variation on [DM94, 2.6].

Proposition 2.5. *Let su be the Jordan decomposition of an element of $(\mathbf{G}^1)^F$ and λ a class function on $(\mathbf{L}^1)^F$;*

(i) *if s is central, if we set*

$$Q_{\mathbf{L}^0}^{\mathbf{G}^0}(u, v) = \begin{cases} \text{Trace}((u, v) \mid H_c^*(\mathbf{Y}_{\mathbf{U}}^0)) & \text{if } uv \in \mathbf{G}^0 \\ 0 & \text{otherwise} \end{cases},$$

we have

$$(R_{\mathbf{L}^1}^{\mathbf{G}^1} \lambda)(su) = |(\mathbf{L}^0)^F|^{-1} \sum_{v \in (\mathbf{L}^0 \cdot u)_{\text{unip}}^F} Q_{\mathbf{L}^0}^{\mathbf{G}^0}(u, v^{-1}) \lambda(sv)$$

(ii) *in general*

$$(R_{\mathbf{L}^1}^{\mathbf{G}^1} \lambda)(su) = \sum_{\{h \in (\mathbf{G}^0)^F \mid {}^h \mathbf{L} \ni s\}} \frac{|{}^h \mathbf{L}^0 \cap C_{\mathbf{G}}(s)^{0F}|}{|(\mathbf{L}^0)^F| |C_{\mathbf{G}}(s)^{0F}|} R_{{}^h \mathbf{L}^1 \cap C_{\mathbf{G}}(s)^{0 \cdot su}}^{C_{\mathbf{G}}(s)^{0 \cdot su}} ({}^h \lambda)(su).$$

(iii) *if tv is the Jordan decomposition of an element of $(\mathbf{L}^1)^F$ and γ a class function on $(\mathbf{G}^1)^F$; we have*

$$(*R_{\mathbf{L}^1}^{\mathbf{G}^1} \gamma)(tv) = |(\mathbf{G}^{t0})^F|^{-1} \sum_{u \in (\mathbf{G}^{t0} \cdot v)_{\text{unip}}^F} Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u, v^{-1}) \gamma(tu).$$

In the above we abused notation to write ${}^h \mathbf{L} \ni s$ for $\langle \mathbf{L}^1 \rangle \ni {}^{h^{-1}} s$.

Proof. (i) results from [DM94, 2.6(i)] using the same arguments as the proof of [DM94, 2.10]; we then get (ii) by plugging back (i) in [DM94, 2.6(i)]. \square

In our setting the Mackey formula [DM94, 3.1] is still valid in the cases where we proved it [DM94, Théorème 3.2] and [DM94, Théorème 4.5]. Before stating it notice that [DM94, 1.40] remains true without assuming that $(\mathbf{G}^1)^F$ contains quasi-central elements, replacing in the proof $(\mathbf{G}^0)^F \cdot \sigma$ with $(\mathbf{G}^1)^F$, which shows that any F -stable “Levi” of \mathbf{G}^1 is $(\mathbf{G}^0)^F$ -conjugate to a “Levi” containing σ . This explains why we only state the Mackey formula in the case of “Levis” containing σ .

Theorem 2.6. *If \mathbf{L}^1 and \mathbf{M}^1 are two F -stable “Levis” of \mathbf{G}^1 containing σ then under one of the following assumptions:*

- \mathbf{L}^0 (resp. \mathbf{M}^0) is a Levi subgroup of an F -stable parabolic subgroup normalized by \mathbf{L}^1 (resp. \mathbf{M}^1).
- one of \mathbf{L}^1 and \mathbf{M}^1 is a “torus”

we have

$$*R_{\mathbf{L}^1}^{\mathbf{G}^1} R_{\mathbf{M}^1}^{\mathbf{G}^1} = \sum_{x \in [\mathbf{L}^{\sigma 0 F} \setminus \mathcal{S}_{\mathbf{G}^{\sigma 0}}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})^F / \mathbf{M}^{\sigma 0 F}]} R_{(\mathbf{L}^1 \cap {}^x \mathbf{M}^1)}^{\mathbf{L}^1} * R_{(\mathbf{L}^1 \cap {}^x \mathbf{M}^1)}^{{}^x \mathbf{M}^1} \text{ad } x$$

where $\mathcal{S}_{\mathbf{G}^{\sigma 0}}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})$ is the set of elements $x \in \mathbf{G}^{\sigma 0}$ such that $\mathbf{L}^{\sigma 0} \cap {}^x \mathbf{M}^{\sigma 0}$ contains a maximal torus of $\mathbf{G}^{\sigma 0}$.

Proof. We first prove the theorem in the case of F -stable parabolic subgroups $\mathbf{P}^0 = \mathbf{L}^0 \ltimes \mathbf{U}$ and $\mathbf{Q}^0 = \mathbf{M}^0 \ltimes \mathbf{V}$ following the proof of [DM94, 3.2]. The difference is that the variety we consider here is the intersection with \mathbf{G}^0 of the variety considered in *loc. cit.*. Here, the left-hand side of the Mackey formula is given by $\overline{\mathbb{Q}}_\ell[(\mathbf{U}^F \backslash (\mathbf{G}^0)^F / \mathbf{V}^F)^\sigma]$ instead of $\overline{\mathbb{Q}}_\ell[(\mathbf{U}^F \backslash (\mathbf{G}^0)^F \cdot \langle \sigma \rangle / \mathbf{V}^F)^\sigma]$. Nevertheless we can use [DM94, Lemma 3.3] which remains valid with the same proof. As for [DM94, Lemma 3.5], we have to replace it with

Lemma 2.7. *For any $x \in \mathcal{S}_{\mathbf{G}^0}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})^F$ the map $(l(\mathbf{L}^0 \cap {}^x \mathbf{V}^F), ({}^x \mathbf{M}^0 \cap \mathbf{U}^F) \cdot {}^x m) \mapsto \mathbf{U}^F l x m \mathbf{V}^F$ is an isomorphism from $(\mathbf{L}^0)^F / (\mathbf{L}^0 \cap {}^x \mathbf{V}^F) \times_{(\mathbf{L}^0 \cap {}^x \mathbf{M}^0)^F} ({}^x \mathbf{M}^0 \cap \mathbf{U}^F) \backslash {}^x (\mathbf{M}^0)^F$ to $\mathbf{U}^F \backslash (\mathbf{P}^0)^F x (\mathbf{Q}^0)^F / \mathbf{V}^F$ which is compatible with the action of $(\mathbf{L}^1)^F \times ((\mathbf{M}^1)^F)^{-1}$ where the action of $(\lambda, \mu^{-1}) \in (\mathbf{L}^1)^F \times ((\mathbf{M}^1)^F)^{-1}$ maps $(l(\mathbf{L}^0 \cap {}^x \mathbf{V}^F), ({}^x \mathbf{M}^0 \cap \mathbf{U}^F) \cdot {}^x m)$ to the class of $(\lambda l \nu^{-1}(\mathbf{L}^0 \cap {}^x \mathbf{V}^F), ({}^x \mathbf{M}^0 \cap \mathbf{U}^F) \cdot \nu {}^x m \mu^{-1})$ with $\nu \in (\mathbf{L}^1)^F \cap {}^x (\mathbf{M}^1)^F$ (independent of ν).*

Proof. The isomorphism of the lemma involves only connected groups and is a known result (see e.g. [DM91, 5.7]). The compatibility with the actions is straightforward. \square

This allows to complete the proof in the first case.

We now prove the second case following section 4 of [DM94]. We first notice that the statement and proof of Lemma 4.1 in [DM94] don't use the element σ but only its action. In Lemma 4.2, 4.3 and 4.4 there is no σ involved but only the action of the groups \mathbf{L}^F and \mathbf{M}^F on the pieces of a variety depending only on \mathbf{L} , \mathbf{M} and the associated parabolics. This gives the second case. \square

We now rephrase [DM94, 4.8] and [DM94, 4.11] in our setting, specializing the Mackey formula to the case of two “tori”. Let \mathcal{T}_1 be the set of “tori” of \mathbf{G}^1 ; if $\mathbf{T}^1 = N_{\mathbf{G}^1}(\mathbf{T}^0, \mathbf{B}^0) \in \mathcal{T}_1^F$ then \mathbf{T}^0 is F -stable. We define $\text{Irr}((\mathbf{T}^1)^F)$ as the set of restrictions to $(\mathbf{T}^1)^F$ of extensions to $\langle (\mathbf{T}^1)^F \rangle$ of elements of $\text{Irr}((\mathbf{T}^0)^F)$.

Proposition 2.8. *If $\mathbf{T}^1, \mathbf{T}'^1 \in \mathcal{T}_1^F$ and $\theta \in \text{Irr}((\mathbf{T}^1)^F), \theta' \in \text{Irr}((\mathbf{T}'^1)^F)$ then*

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}'^1}^{\mathbf{G}^1}(\theta') \rangle_{(\mathbf{G}^1)^F} = 0 \text{ unless } (\mathbf{T}^1, \theta) \text{ and } (\mathbf{T}'^1, \theta') \text{ are } (\mathbf{G}^0)^F\text{-conjugate.}$$

And

- (i) *If for some $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)$ and $\zeta \neq 1$ we have ${}^n \theta = \zeta \theta$ then $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$.*
- (ii) *Otherwise $\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F} = |\{n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) \mid {}^n \theta = \theta\}| / |(\mathbf{T}^1)^F|$.*

If $\mathbf{T}^1 = \mathbf{T}'^1$ the above can be written

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta') \rangle_{(\mathbf{G}^1)^F} = \langle \text{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \theta, \text{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \theta' \rangle_{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}$$

where when $A^1 \subset B^1$ are cosets of finite groups $A^0 \subset B^0$ and χ is a A^0 -class function on A^1 we set $\text{Ind}_{A^1}^{B^1} \chi(x) = |A^0|^{-1} \sum_{\{y \in B^0 \mid yx \in A^1\}} \chi(yx)$.

Proof. As noticed above Theorem 2.6 we may assume that \mathbf{T}^1 and \mathbf{T}'^1 contain σ . By [DM94, 1.39], if \mathbf{T}^1 and \mathbf{T}'^1 contain σ , they are $(\mathbf{G}^0)^F$ conjugate if and only if they are conjugate under $\mathbf{G}^{\sigma 0}$. The Mackey formula shows then that the scalar product vanishes when \mathbf{T}^1 and \mathbf{T}'^1 are not $(\mathbf{G}^0)^F$ -conjugate.

Otherwise we may assume $\mathbf{T}^1 = \mathbf{T}'^1$ and the Mackey formula gives

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F} = |(\mathbf{T}^{\sigma 0})^F|^{-1} \sum_{n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F} \langle \theta, {}^n \theta \rangle_{(\mathbf{T}^1)^F}.$$

The term $\langle \theta, {}^n\theta \rangle_{(\mathbf{T}^1)^F}$ is 0 unless ${}^n\theta = \zeta_n \theta$ for some constant ζ_n and in this last case $\langle \theta, {}^n\theta \rangle_{(\mathbf{T}^1)^F} = \bar{\zeta}_n$. If ${}^{n'}\theta = \zeta_{n'} \theta$ then ${}^{nn'}\theta = \zeta_{n'} {}^n\theta = \zeta_{n'} \zeta_n \theta$ thus the ζ_n form a group; if this group is not trivial, that is some ζ_n is not equal to 1, we have $\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F} = 0$ which implies that in this case $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$. This gives (i) since by [DM94, 1.39], if $\mathbf{T}^1 \ni \sigma$ then $N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) = N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F \cdot (\mathbf{T}^0)^F$, so that if there exists n as in (i) there exists an $n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F$ with same action on θ since $(\mathbf{T}^0)^F$ has trivial action on θ .

In case (ii), for each non-zero term we have ${}^n\theta = \theta$ and we have to check that the value $|((\mathbf{T}^0)^F)^{-1} \{n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F \mid {}^n\theta = \theta\}|$ given by the Mackey formula is equal to the stated value. This results again from [DM94, 1.39] written $N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) = N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^1)^F \cdot (\mathbf{T}^0)^F$, and from $N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^1)^F \cap (\mathbf{T}^0)^F = ((\mathbf{T}^0)^F)^F$.

We now prove the final remark. By definition we have

$$\begin{aligned} & \langle \text{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \theta, \text{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \theta' \rangle_{N_{\mathbf{G}^1}(\mathbf{T}^0)^F} = \\ & |N_{\mathbf{G}^1}(\mathbf{T}^0)^F|^{-1} |(\mathbf{T}^1)^F|^{-2} \sum_{x \in N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \sum_{\{n, n' \in N_{\mathbf{G}^1}(\mathbf{T}^0)^F \mid {}^n x, {}^{n'} x \in \mathbf{T}^1\}} \theta({}^n x) \overline{\theta({}^{n'} x)}. \end{aligned}$$

Doing the summation over $t = {}^n x$ and $n'' = n' n^{-1} \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ we get

$$|N_{\mathbf{G}^1}(\mathbf{T}^0)^F|^{-1} |(\mathbf{T}^1)^F|^{-2} \sum_{t \in (\mathbf{T}^1)^F} \sum_{n \in N_{\mathbf{G}^1}(\mathbf{T}^0)^F} \sum_{\{n'' \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F \mid n'' t \in \mathbf{T}^1\}} \theta(t) \overline{\theta(n'' t)}.$$

The conditions $n'' \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ together with $n'' t \in \mathbf{T}^1$ are equivalent to $n'' \in N_{\mathbf{G}^0}(\mathbf{T}^1)^F$, so that we get $|(\mathbf{T}^1)^F|^{-1} \sum_{n'' \in N_{\mathbf{G}^0}(\mathbf{T}^1)^F} \langle \theta, {}^{n''}\theta \rangle_{(\mathbf{T}^1)^F}$. As explained in the first part of the proof, the scalar product $\langle \theta, {}^{n''}\theta \rangle_{(\mathbf{T}^1)^F}$ is zero unless ${}^{n''}\theta = \zeta_{n''} \theta$ for some root of unity $\zeta_{n''}$ and arguing as in the first part of the proof we find that the above sum is zero if there exists n'' such that $\zeta_{n''} \neq 1$ and is equal to $|(\mathbf{T}^1)^F|^{-1} |\{n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) \mid {}^n\theta = \theta\}|$ otherwise. \square

Remark 2.9. In the context of Proposition 2.8, if σ is F -stable then we may apply θ to it and for any $n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F$ we have $\theta({}^n\sigma) = \theta(\sigma)$ so for any $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)$ and ζ such that ${}^n\theta = \zeta \theta$ we have $\zeta = 1$. When $H^1(F, Z\mathbf{G}^0) = 1$ we may choose σ to be F -stable, so that $\zeta \neq 1$ never happens.

Here is an example where $\zeta_n = -1$, thus $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$: we take again the context of Example 2.1 and take $\mathbf{T}^0 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and let $\mathbf{T}^1 = \mathbf{T}^0 \cdot s$; let us define θ on $ts \in (\mathbf{T}^1)^F$ by $\theta(ts) = -\lambda(t)$ where λ is the non-trivial order 2 character of $(\mathbf{T}^0)^F$ (Legendre symbol); then for any $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) \setminus \mathbf{T}^0$ we have ${}^n\theta = -\theta$. \square

Proposition [DM94, 4.11] extends as follows to our context:

Corollary 2.10 (of 2.8). *Let $p^{\mathbf{G}^1}$ be the projector to uniform functions on $(\mathbf{G}^1)^F$. We have*

$$p^{\mathbf{G}^1} = |(\mathbf{G}^1)^F|^{-1} \sum_{\mathbf{T}^1 \in \mathcal{T}_1^F} |(\mathbf{T}^1)^F| R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ {}^* R_{\mathbf{T}^1}^{\mathbf{G}^1}.$$

Proof. We have only to check that for any $\theta \in \text{Irr}((\mathbf{T}^1)^F)$ such that $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \neq 0$ and any class function χ on $(\mathbf{G}^1)^F$ we have $\langle p^{\mathbf{G}^1} \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F} = \langle \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F}$.

By Proposition 2.8, to evaluate the left-hand side we may restrict the sum to tori conjugate to \mathbf{T}^1 , so we get

$$\begin{aligned} \langle p^{\mathbf{G}^1} \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F} &= |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|^{-1} |(\mathbf{T}^1)^F| \langle R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ {}^* R_{\mathbf{T}^1}^{\mathbf{G}^1} \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F} \\ &= |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|^{-1} |(\mathbf{T}^1)^F| \langle \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ {}^* R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) \rangle_{(\mathbf{G}^1)^F}. \end{aligned}$$

The equality to prove is true if $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$; otherwise by Proposition 2.8 we have ${}^* R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = |(\mathbf{T}^1)^F|^{-1} \sum_{n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)} {}^n \theta$, whence in that case

$$R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ {}^* R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = |(\mathbf{T}^1)^F|^{-1} |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)| R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta),$$

since $R_{\mathbf{T}^1}^{\mathbf{G}^1}({}^n \theta) = R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)$, whence the result. \square

We now adapt the definition of duality to our setting.

Definition 2.11. • For a connected reductive group \mathbf{G} , we define the \mathbb{F}_q -rank as the maximal dimension of a split torus, and define $\varepsilon_{\mathbf{G}} = (-1)^{\mathbb{F}_q\text{-rank of } \mathbf{G}}$ and $\eta_{\mathbf{G}} = \varepsilon_{\mathbf{G}} / \text{rad } \mathbf{G}$.
• For an F -stable connected component \mathbf{G}^1 of a (possibly disconnected) reductive group we define $\varepsilon_{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^{\sigma_0}}$ and $\eta_{\mathbf{G}^1} = \eta_{\mathbf{G}^{\sigma_0}}$ where σ is a quasi-central element of \mathbf{G}^1 which induces an F -stable automorphism of \mathbf{G}^0 .

Let us see that these definitions agree with [DM94]: in [DM94, 3.6(i)], we define $\varepsilon_{\mathbf{G}^1}$ to be $\varepsilon_{\mathbf{G}^0 \tau}$ where τ is any quasi-semi-simple element of \mathbf{G}^1 which induces an F -stable automorphism of \mathbf{G}^0 and lies in a “torus” of the form $N_{\mathbf{G}^1}(\mathbf{T}_0 \subset \mathbf{B}_0)$ where both \mathbf{T}^0 and \mathbf{B}^0 are F -stable; by [DM94, 1.36(ii)] a σ as above is such a τ .

We fix an F -stable pair $(\mathbf{T}_0 \subset \mathbf{B}_0)$ and define duality on $\text{Irr}((\mathbf{G}^1)^F)$ by

$$(2.12) \quad D_{\mathbf{G}^1} = \sum_{\mathbf{P}^0 \supset \mathbf{B}^0} \eta_{\mathbf{L}^1} R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ {}^* R_{\mathbf{L}^1}^{\mathbf{G}^1}$$

where in the sum \mathbf{P}^0 runs over F -stable parabolic subgroups containing \mathbf{B}^0 such that $N_{\mathbf{G}^1}(\mathbf{P}^0)$ is non empty, and \mathbf{L}^1 denotes $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ where \mathbf{L}^0 is the Levi subgroup of \mathbf{P}^0 containing \mathbf{T}^0 . The duality thus defined coincides with the duality defined in [DM94, 3.10] when σ is in $(\mathbf{G}^1)^F$.

In our context we can define $\text{St}_{\mathbf{G}^1}$ similarly to [DM94, 3.16], as $D_{\mathbf{G}^1}(\text{Id}_{\mathbf{G}^1})$, and [DM94, 3.18] remains true:

Proposition 2.13. $\text{St}_{\mathbf{G}^1}$ vanishes outside quasi-semi-simple elements, and if $x \in (\mathbf{G}^1)^F$ is quasi-semi-simple we have

$$\text{St}_{\mathbf{G}^1}(x) = \varepsilon_{\mathbf{G}^1} \varepsilon_{(\mathbf{G}^x)^0} |(\mathbf{G}^x)^0|_p.$$

3. A GLOBAL FORMULA FOR THE SCALAR PRODUCT OF DELIGNE-LUSZTIG CHARACTERS.

In this section we give a result of a different flavor, where we do not restrict our attention to a connected component \mathbf{G}^1 .

Definition 3.1. For any character θ of \mathbf{T}^F , we define $R_{\mathbf{T}}^{\mathbf{G}}$ as in [DM94, 2.2]. If for a “torus” \mathbf{T} and $\alpha = g\mathbf{G}^0 \in \mathbf{G}/\mathbf{G}^0$ we denote by $\mathbf{T}^{[\alpha]}$ or $\mathbf{T}^{[g]}$ the unique connected component of \mathbf{T} which meets $g\mathbf{G}^0$, this is equivalent for $g \in \mathbf{G}^F$ to

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = |(\mathbf{T}^0)^F|/|\mathbf{T}^F| \sum_{\{a \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid {}^a g \in \mathbf{T}^F(\mathbf{G}^0)^F\}} R_{\mathbf{T}^{[a_g]}}^{\mathbf{G}^{[a_g]}}(\theta)({}^a g)$$

where the right-hand side is defined by 2.4 (see [DM94, 2.3]).

We deduce from Proposition 2.8 the following formula for the whole group \mathbf{G} :

Proposition 3.2. *Let \mathbf{T}, \mathbf{T}' be two “tori” of \mathbf{G} and let $\theta \in \text{Irr}(\mathbf{T}^F), \theta' \in \text{Irr}(\mathbf{T}'^F)$. Then $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta') \rangle_{\mathbf{G}^F} = 0$ if \mathbf{T}^0 and \mathbf{T}'^0 are not \mathbf{G}^F -conjugate, and if $\mathbf{T}^0 = \mathbf{T}'^0$ we have*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta') \rangle_{\mathbf{G}^F} = \langle \text{Ind}_{\mathbf{T}^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta), \text{Ind}_{\mathbf{T}'^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta') \rangle_{N_{\mathbf{G}}(\mathbf{T}^0)^F}.$$

Proof. Definition 3.1 can be written

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = |(\mathbf{T}^0)^F|/|\mathbf{T}^F| \sum_{\{a \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid a g \in \mathbf{T}^F(\mathbf{G}^0)^F\}} R_{(a^{-1}\mathbf{T})^{[\alpha]}}^{\mathbf{G}}(a^{-1}\theta)(g).$$

So the scalar product we want to compute is equal to

$$\begin{aligned} \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta') \rangle_{\mathbf{G}^F} &= \frac{1}{|\mathbf{G}^F|} \frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|} \frac{|\mathbf{T}'^0|^F}{|\mathbf{T}'^F|} \\ &\sum_{\substack{\alpha \in \mathbf{G}^F/\mathbf{G}^0{}^F \\ g \in (\mathbf{G}^0)^F \cdot \alpha}} \sum_{\substack{\{a \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid a\alpha \in \mathbf{T}^F(\mathbf{G}^0)^F\} \\ \{a' \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid a'\alpha \in \mathbf{T}'^F(\mathbf{G}^0)^F\}}} R_{(a^{-1}\mathbf{T})^{[\alpha]}}^{\mathbf{G}, \alpha}(a^{-1}\theta)(g) \overline{R_{(a'^{-1}\mathbf{T}')^{[\alpha]}}^{\mathbf{G}, \alpha}(a'^{-1}\theta')(g)}, \end{aligned}$$

which can be written

$$\begin{aligned} \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta') \rangle_{\mathbf{G}^F} &= \frac{|(\mathbf{G}^0)^F|}{|\mathbf{G}^F|} \frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|} \frac{|\mathbf{T}'^0|^F}{|\mathbf{T}'^F|} \\ &\sum_{\alpha \in \mathbf{G}^F/\mathbf{G}^0{}^F} \sum_{\substack{\{a \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid a\alpha \in \mathbf{T}^F(\mathbf{G}^0)^F\} \\ \{a' \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid a'\alpha \in \mathbf{T}'^F(\mathbf{G}^0)^F\}}} \langle R_{(a^{-1}\mathbf{T})^{[\alpha]}}^{\mathbf{G}, \alpha}(a^{-1}\theta), R_{(a'^{-1}\mathbf{T}')^{[\alpha]}}^{\mathbf{G}, \alpha}(a'^{-1}\theta') \rangle_{(\mathbf{G}^0)^F \cdot \alpha}. \end{aligned}$$

By Proposition 2.8 the scalar product on the right-hand side is zero unless $(a^{-1}\mathbf{T})^{[\alpha]}$ and $(a'^{-1}\mathbf{T}')^{[\alpha]}$ are $(\mathbf{G}^0)^F$ -conjugate, which implies that \mathbf{T}^0 and \mathbf{T}'^0 are $(\mathbf{G}^0)^F$ -conjugate. So we can assume that $\mathbf{T}^0 = \mathbf{T}'^0$. Moreover for each a' indexing a non-zero summand, there is a representative $y \in a'^{-1}(\mathbf{G}^0)^F$ such that $(y\mathbf{T}')^{[\alpha]} = (a^{-1}\mathbf{T})^{[\alpha]}$. This last equality and the condition on a imply the condition $a'\alpha \in \mathbf{T}'^F(\mathbf{G}^0)^F$ since this condition can be written $(y\mathbf{T}')^{[\alpha]} \neq \emptyset$. Thus we can do the summation over all such $y \in \mathbf{G}^F$, provided we divide by $|N_{(\mathbf{G}^0)^F}((a^{-1}\mathbf{T})^{[\alpha]})|$. So we get, applying Proposition 2.8 that the above expression is equal to

$$\begin{aligned} &\frac{|(\mathbf{G}^0)^F|}{|\mathbf{G}^F|} \frac{|(\mathbf{T}^0)^F|^2}{|\mathbf{T}^F||\mathbf{T}'^F|} \sum_{\alpha \in \mathbf{G}^F/\mathbf{G}^0{}^F} \sum_{\{a \in [\mathbf{G}^F/(\mathbf{G}^0)^F] \mid a\alpha \in \mathbf{T}^F(\mathbf{G}^0)^F\}} |N_{(\mathbf{G}^0)^F}((a^{-1}\mathbf{T})^{[\alpha]})|^{-1} \\ &\sum_{\{y \in \mathbf{G}^F \mid (y\mathbf{T}')^{[\alpha]} = (a^{-1}\mathbf{T})^{[\alpha]}\}} \langle \text{Ind}_{(a^{-1}\mathbf{T})^{[\alpha]}}^{N_{\mathbf{G}^0, \alpha}(\mathbf{T}^0)^F} a^{-1}\theta, \text{Ind}_{(a^{-1}\mathbf{T})^{[\alpha]}}^{N_{\mathbf{G}^0, \alpha}(\mathbf{T}^0)^F} y\theta' \rangle_{N_{\mathbf{G}^0, \alpha}(\mathbf{T}^0)^F}. \end{aligned}$$

We now conjugate everything by a , take ay as new variable y and set $b = a\alpha$. We get

$$\begin{aligned} (3.3) \quad &\frac{|(\mathbf{T}^0)^F|^2}{|\mathbf{T}^F||\mathbf{T}'^F|} \sum_{b \in \mathbf{T}^F/(\mathbf{T}^0)^F} |N_{(\mathbf{G}^0)^F}(\mathbf{T}^{[b]})|^{-1} \\ &\sum_{\{y \in \mathbf{G}^F \mid (y\mathbf{T}')^{[b]} = \mathbf{T}^{[b]}\}} \langle \text{Ind}_{\mathbf{T}^{[b]}{}^F}^{N_{\mathbf{G}^0, b}(\mathbf{T}^0)^F} \theta, \text{Ind}_{\mathbf{T}^{[b]}{}^F}^{N_{\mathbf{G}^0, b}(\mathbf{T}^0)^F} y\theta' \rangle_{N_{\mathbf{G}^0, b}(\mathbf{T}^0)^F}, \end{aligned}$$

since for $b \in \mathbf{T}^F/(\mathbf{T}^0)^F$ any choice of $a \in \mathbf{G}^F/(\mathbf{G}^0)^F$ gives an $\alpha = a^{-1}b$ which satisfies the condition ${}^a\alpha \in \mathbf{T}^F/(\mathbf{G}^0)^F$.

Let us now transform the right-hand side of 3.2. Using the definition we have

$$\begin{aligned}
& \langle \text{Ind}_{\mathbf{T}^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta), \text{Ind}_{\mathbf{T}'^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta') \rangle_{N_{\mathbf{G}}(\mathbf{T}^0)^F} = \\
& \quad \frac{|\mathbf{T}^F|^{-1}|\mathbf{T}'^F|^{-1}|N_{\mathbf{G}}(\mathbf{T}^0)^F|^{-1}}{\sum_{\{n, x, x' \in N_{\mathbf{G}}(\mathbf{T}^0)^F \mid x_n \in \mathbf{T}, x'_n \in \mathbf{T}'\}}} \theta(x_n) \overline{\theta'(x'_n)} = \\
& \quad \frac{|\mathbf{T}^F|^{-1}|\mathbf{T}'^F|^{-1}|N_{\mathbf{G}}(\mathbf{T}^0)^F|^{-1}}{\sum_{b, a, a' \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]} \sum_{\left\{ \begin{array}{l} n \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F b \\ x_0, x'_0 \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F \end{array} \middle| \begin{array}{l} x_0 n \in ({}^{a^{-1}}\mathbf{T})^{[b]} \\ x'_0 n \in ({}^{a'^{-1}}\mathbf{T}')^{[b]} \end{array} \right\}}} a^{-1} \theta(x_0 n) \overline{a'^{-1} \theta'(x'_0 n)} = \\
& \quad \frac{|\mathbf{T}^0|^F}{|\mathbf{T}^F|} \frac{|\mathbf{T}'^0|^F}{|\mathbf{T}'^F|} \frac{|N_{\mathbf{G}^0}(\mathbf{T}^0)^F|}{|N_{\mathbf{G}}(\mathbf{T}^0)^F|} \\
& \quad \sum_{b, a, a' \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]} \langle \text{Ind}_{({}^{a^{-1}}\mathbf{T})^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b \cdot a^{-1}} \theta, \text{Ind}_{({}^{a'^{-1}}\mathbf{T}')^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b \cdot a'^{-1}} \theta' \rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F b}.
\end{aligned}$$

We may simplify the sum by conjugating by a the terms in the scalar product to get

$$\langle \text{Ind}_{\mathbf{T}^{[a]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot {}^a b} \theta, \text{Ind}_{({}^{aa'^{-1}}\mathbf{T}')^{[a]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot {}^a b \cdot aa'^{-1}} \theta' \rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F {}^a b}$$

then we may take, given a , the conjugate ${}^a b$ as new variable b , and aa'^{-1} as the new variable a' to get

$$\frac{|\mathbf{T}^0|^F}{|\mathbf{T}^F|} \frac{|\mathbf{T}'^0|^F}{|\mathbf{T}'^F|} \sum_{b, a' \in [\frac{N_{\mathbf{G}}(\mathbf{T}^0)^F}{N_{\mathbf{G}^0}(\mathbf{T}^0)^F}]} \langle \text{Ind}_{\mathbf{T}^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} \theta, \text{Ind}_{({}^{a'}\mathbf{T}')^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b \cdot a'} \theta' \rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F b}.$$

Now, by Frobenius reciprocity, for the inner scalar product not to vanish, there must be some element $x \in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ such that $x({}^{a'}\mathbf{T}')^{[b]}F$ meets $\mathbf{T}^{[b]}F$ which, considering the definitions, implies that $(x a' \mathbf{T}')^{[b]} = \mathbf{T}^{[b]}$. We may then conjugate the term $\text{Ind}_{({}^{a'}\mathbf{T}')^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b \cdot a' \theta'}$ by such an x to get $\text{Ind}_{\mathbf{T}^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b \cdot x a' \theta'}$ and take $y = x a'$ as a new variable, provided we count the number of x for a given a' , which is $|N_{\mathbf{G}^0}(\mathbf{T}^{[b]})^F|$. We get

$$\begin{aligned}
(3.4) \quad & \frac{|\mathbf{T}^0|^F}{|\mathbf{T}^F|} \frac{|\mathbf{T}'^0|^F}{|\mathbf{T}'^F|} \sum_{b \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]} |N_{\mathbf{G}^0}(\mathbf{T}^{[b]})^F|^{-1} \\
& \sum_{\{y \in N_{\mathbf{G}}(\mathbf{T}^0)^F \mid (y \mathbf{T}')^{[b]} = \mathbf{T}^{[b]}\}} \langle \text{Ind}_{\mathbf{T}^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b} \theta, \text{Ind}_{\mathbf{T}^{[b]}F}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F \cdot b \cdot y \theta'} \rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F b}.
\end{aligned}$$

Since any $b \in [N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]$ such that $\mathbf{T}^{[b]}F$ is not empty has a representative in \mathbf{T}^F we can do the first summation over $b \in [\mathbf{T}^F/(\mathbf{T}^0)^F]$ so that 3.3 is equal to 3.4. \square

4. COUNTING UNIPOTENT ELEMENTS IN DISCONNECTED GROUPS

Proposition 4.1. *Assume $\mathbf{G}^1/\mathbf{G}^0$ unipotent and take $\sigma \in \mathbf{G}^1$ unipotent F -stable and quasi-central (see 2.2). Then the number of unipotent elements of $(\mathbf{G}^1)^F$ is given by $|(\mathbf{G}^{\sigma^0})^F|_p^2 |(\mathbf{G}^0)^F| / |(\mathbf{G}^{\sigma^0})^F|$.*

Proof. Let χ_U be the characteristic function of the set of unipotent elements of $(\mathbf{G}^1)^F$. Then $|(\mathbf{G}^1)^F|_{\text{unip}} = |(\mathbf{G}^1)^F| \langle \chi_U, \text{Id} \rangle_{(\mathbf{G}^1)^F}$ and

$$\langle \chi_U, \text{Id} \rangle_{(\mathbf{G}^1)^F} = \langle D_{\mathbf{G}^1}(\chi_U), D_{\mathbf{G}^1}(\text{Id}) \rangle_{(\mathbf{G}^1)^F} = \langle D_{\mathbf{G}^1}(\chi_U), \text{St}_{\mathbf{G}^1} \rangle_{(\mathbf{G}^1)^F},$$

the first equality since $D_{\mathbf{G}^1}$ is an isometry by [DM94, 3.12]. According to [DM94, 2.11], for any σ -stable and F -stable Levi subgroup \mathbf{L}^0 of a σ -stable parabolic subgroup of \mathbf{G}^0 , setting $\mathbf{L}^1 = \mathbf{L}^0 \cdot \sigma$, we have $R_{\mathbf{L}^1}^{\mathbf{G}^1}(\pi \cdot \chi_U|_{(\mathbf{L}^1)^F}) = R_{\mathbf{L}^1}^{\mathbf{G}^1}(\pi) \cdot \chi_U$ and $*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\varphi) \cdot \chi_U|_{(\mathbf{L}^1)^F} = *R_{\mathbf{L}^1}^{\mathbf{G}^1}(\varphi \cdot \chi_U)$, thus, by 2.12, $D_{\mathbf{G}^1}(\pi \cdot \chi_U) = D_{\mathbf{G}^1}(\pi) \cdot \chi_U$; in particular $D_{\mathbf{G}^1}(\chi_U) = D_{\mathbf{G}^1}(\text{Id}) \cdot \chi_U = \text{St}_{\mathbf{G}^1} \cdot \chi_U$. Now, by Proposition 2.13, the only unipotent elements on which $\text{St}_{\mathbf{G}^1}$ does not vanish are the quasi-semi-simple (thus quasi-central) ones; by [DM94, 1.37] all such are in the \mathbf{G}^{0^F} -class of σ and, again by 2.13 we have $\text{St}_{\mathbf{G}^1}(\sigma) = |(\mathbf{G}^{\sigma^0})^F|_p$. We get

$$\begin{aligned} |(\mathbf{G}^1)^F| \langle D_{\mathbf{G}^1} \chi_U, \text{St}_{\mathbf{G}^1} \rangle_{(\mathbf{G}^1)^F} &= |(\mathbf{G}^1)^F| \langle \text{St}_{\mathbf{G}^1} \cdot \chi_U, \text{St}_{\mathbf{G}^1} \rangle_{(\mathbf{G}^1)^F} \\ &= |\{\mathbf{G}^{0^F}\text{-class of } \sigma\}| |(\mathbf{G}^{\sigma^0})^F|_p^2 \end{aligned}$$

whence the proposition. \square

Example 4.2. The formula of Proposition 4.1 applies in the following cases where σ induces a diagram automorphism of order 2 and q is a power of 2:

- $\mathbf{G}^0 = \text{SO}_{2n}$, $(\mathbf{G}^{\sigma^0})^F = \text{SO}_{2n-1}(\mathbb{F}_q)$;
- $\mathbf{G}^0 = \text{GL}_{2n}$, $(\mathbf{G}^{\sigma^0})^F = \text{Sp}_{2n}(\mathbb{F}_q)$;
- $\mathbf{G}^0 = \text{GL}_{2n+1}$, $(\mathbf{G}^{\sigma^0})^F = \text{SO}_{2n+1}(\mathbb{F}_q) \simeq \text{Sp}_{2n}(\mathbb{F}_q)$;
- $\mathbf{G}^0 = E_6$, $(\mathbf{G}^{\sigma^0})^F = F_4(\mathbb{F}_q)$;

And it applies to the case where $\mathbf{G}^0 = \text{SO}_8$ where σ induces a diagram automorphism of order 3 and q is a power of 3, in which case $(\mathbf{G}^{\sigma^0})^F = G_2(\mathbb{F}_q)$.

5. TENSORING BY THE STEINBERG CHARACTER

Proposition 5.1. *Let \mathbf{L}^1 be an F -stable “Levi” of \mathbf{G}^1 . Then, for any class function γ on $(\mathbf{G}^1)^F$ we have:*

$$*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma \cdot \varepsilon_{\mathbf{G}^1} \text{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{L}^1} \text{St}_{\mathbf{L}^1} \text{Res}_{(\mathbf{L}^1)^F}^{(\mathbf{G}^1)^F} \gamma.$$

Proof. Let su be the Jordan decomposition of a quasi-semi-simple element of \mathbf{G}^1 with s semisimple. We claim that u is quasi-central in \mathbf{G}^s . Indeed su , being quasi-semi-simple, is in a “torus” \mathbf{T} , thus s and u also are in \mathbf{T} . By [DM94, 1.8(iii)] the intersection of $\mathbf{T} \cap \mathbf{G}^s$ is a “torus” of \mathbf{G}^s , thus u is quasi-semi-simple in \mathbf{G}^s , hence quasi-central since unipotent.

Let tv be the Jordan decomposition of an element $l \in (\mathbf{L}^1)^F$ where t is semisimple. Since $\text{St}_{\mathbf{L}^1}$ vanishes outside quasi-semi-simple elements the right-hand side of the proposition vanishes on l unless it is quasi-semi-simple which by our claim

means that v is quasi-central in \mathbf{L}^t . By the character formula 2.5 the left-hand side of the proposition evaluates at l to

$$*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma \cdot \varepsilon_{\mathbf{G}^1} \text{St}_{\mathbf{G}^1})(l) = |(\mathbf{G}^{t0})^F|^{-1} \sum_{u \in (\mathbf{G}^{t0} \cdot v)_{\text{unip}}^F} Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u, v^{-1}) \gamma(tu) \varepsilon_{\mathbf{G}^1} \text{St}_{\mathbf{G}^1}(tu).$$

By the same argument as above, applied to $\text{St}_{\mathbf{G}^1}$, the only non zero terms in the above sum are for u quasi-central in \mathbf{G}^t . For such u , by [DM94, 4.16], $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u, v^{-1})$ vanishes unless u and v are $(\mathbf{G}^{t0})^F$ -conjugate. Hence both sides of the equality to prove vanish unless u and v are quasi-central and $(\mathbf{G}^{t0})^F$ -conjugate. In that case by [DM94, 4.16] and [DM91, (**) page 98] we have $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u, v^{-1}) = Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(1, 1) = \varepsilon_{\mathbf{G}^{t0}} \varepsilon_{\mathbf{L}^{t0}} |(\mathbf{G}^{t0})^F|_{p'} |(\mathbf{L}^{t0})^F|_p$. Taking into account that the $(\mathbf{G}^{t0})^F$ -class of v has cardinality $|(\mathbf{G}^{t0})^F|/|(\mathbf{G}^{t0})^F|$ and that by 2.13 we have $\text{St}_{\mathbf{G}^1}(l) = \varepsilon_{\mathbf{G}^1} \varepsilon_{\mathbf{G}^{t0}} |(\mathbf{G}^{t0})^F|_p$, the left-hand side of the proposition reduces to $\gamma(l) \varepsilon_{\mathbf{L}^{t0}} |(\mathbf{L}^{t0})^F|_p$, which is also the value of the right-hand side by applying 2.13 in \mathbf{L}^1 . \square

By adjunction, we get

Corollary 5.2. *For any class function λ on $(\mathbf{L}^1)^F$ we have:*

$$R_{\mathbf{L}^1}^{\mathbf{G}^1}(\lambda) \varepsilon_{\mathbf{G}^1} \text{St}_{\mathbf{G}^1} = \text{Ind}_{(\mathbf{L}^1)^F}^{(\mathbf{G}^1)^F} (\varepsilon_{\mathbf{L}^1} \text{St}_{\mathbf{L}^1} \lambda)$$

6. CHARACTERISTIC FUNCTIONS OF QUASI-SEMI-SIMPLE CLASSES

One of the goals of this section is Proposition 6.4 where we give a formula for the characteristic function of a quasi-semi-simple class which shows in particular that it is uniform; this generalizes the case of quasi-central elements given in [DM94, 4.14].

If $x \in (\mathbf{G}^1)^F$ has Jordan decomposition $x = su$ we will denote by d_x the map from class functions on $(\mathbf{G}^1)^F$ to class functions on $(C_{\mathbf{G}}(s)^0 \cdot u)^F$ given by

$$(d_x f)(v) = \begin{cases} f(sv) & \text{if } v \in (C_{\mathbf{G}}(s)^0 \cdot u)^F \text{ is unipotent} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 6.1. *Let \mathbf{L}^1 be an F -stable “Levi” of \mathbf{G}^1 . If $x = su$ is the Jordan decomposition of an element of $(\mathbf{L}^1)^F$ we have $d_x \circ *R_{\mathbf{L}^1}^{\mathbf{G}^1} = *R_{C_{\mathbf{L}}(s)^0 \cdot u}^{C_{\mathbf{G}}(s)^0 \cdot u} \circ d_x$.*

Proof. For v unipotent in $(C_{\mathbf{G}}(s)^0 \cdot u)^F$ we have

$$(d_x *R_{\mathbf{L}^1}^{\mathbf{G}^1} f)(v) = (*R_{\mathbf{L}^1}^{\mathbf{G}^1} f)(sv) = (*R_{C_{\mathbf{L}}(s)^0 \cdot su}^{C_{\mathbf{G}}(s)^0 \cdot su} f)(sv) = (*R_{C_{\mathbf{L}}(s)^0 \cdot u}^{C_{\mathbf{G}}(s)^0 \cdot u} d_x f)(v)$$

where the second equality is [DM94, 2.9] and the last is by the character formula 2.5(iii). \square

Proposition 6.2. *If $x = su$ is the Jordan decomposition of an element of $(\mathbf{G}^1)^F$, we have $d_x \circ p^{\mathbf{G}^1} = p^{C_{\mathbf{G}}(s)^0 \cdot u} \circ d_x$.*

Proof. Let f be a class function on $(\mathbf{G}^1)^F$. For $v \in (C_{\mathbf{G}}(s)^0 \cdot u)^F$ unipotent, we have, where the last equality is by 2.10:

$$(d_x p^{\mathbf{G}^1} f)(v) = p^{\mathbf{G}^1} f(sv) = |(\mathbf{G}^1)^F|^{-1} \sum_{\mathbf{T}^1 \in \mathcal{T}_1^F} |(\mathbf{T}^1)^F| (R_{\mathbf{T}^1}^{\mathbf{G}^1} \circ *R_{\mathbf{T}^1}^{\mathbf{G}^1} f)(sv)$$

which by Proposition 2.5(ii) is:

$$\sum_{\mathbf{T}^1 \in \mathcal{T}_1^F} \sum_{\{h \in (\mathbf{G}^0)^F \mid {}^h\mathbf{T} \ni s\}} \frac{|{}^h\mathbf{T}^0 \cap C_{\mathbf{G}}(s)^{0F}|}{|(\mathbf{G}^0)^F| |C_{\mathbf{G}}(s)^{0F}|} (R_{{}^h\mathbf{T} \cap C_{\mathbf{G}}(s)^0 \cdot su}^{C_{\mathbf{G}}(s)^0 \cdot su} \circ {}^h R_{\mathbf{T}^1}^{\mathbf{G}^1} f)(sv).$$

Using that ${}^h R_{\mathbf{T}^1}^{\mathbf{G}^1} f = {}^* R_{{}^h\mathbf{T}^1}^{\mathbf{G}^1} f$ and summing over the ${}^h\mathbf{T}^1$, this becomes

$$\sum_{\{\mathbf{T}^1 \in \mathcal{T}_1^F \mid \mathbf{T} \ni s\}} \frac{|\mathbf{T}^0 \cap C_{\mathbf{G}}(s)^{0F}|}{|C_{\mathbf{G}}(s)^{0F}|} (R_{\mathbf{T}^1 \cap C_{\mathbf{G}}(s)^0 \cdot su}^{C_{\mathbf{G}}(s)^0 \cdot su} \circ {}^* R_{\mathbf{T}^1}^{\mathbf{G}^1} f)(sv).$$

Using that by Proposition 2.5(i) for any class function χ on $\mathbf{T}^1 \cap C_{\mathbf{G}}(s)^0 \cdot su^F$

$$\begin{aligned} (R_{\mathbf{T}^1 \cap C_{\mathbf{G}}(s)^0 \cdot su}^{C_{\mathbf{G}}(s)^0 \cdot su} \chi)(sv) &= |\mathbf{T}^0 \cap C_{\mathbf{G}}(s)^{0F}|^{-1} \sum_{v' \in (\mathbf{T} \cap C_{\mathbf{G}}(s)^0 \cdot u)^F_{\text{unip}}} Q_{(\mathbf{T}^s)^0}^{\mathbf{G}^s}(v, v'^{-1}) \chi(sv') \\ &= R_{\mathbf{T} \cap C_{\mathbf{G}}(s)^0 \cdot u}^{C_{\mathbf{G}}(s)^0 \cdot u} (d_x \chi)(v), \end{aligned}$$

and using Lemma 6.1, we get

$$|C_{\mathbf{G}}(s)^0 \cdot su^F|^{-1} \sum_{\{\mathbf{T}^1 \in \mathcal{T}_1^F \mid \mathbf{T} \ni s\}} |(\mathbf{T}^s)^{0F}| (R_{\mathbf{T} \cap C_{\mathbf{G}}(s)^0 \cdot u}^{C_{\mathbf{G}}(s)^0 \cdot u} \circ {}^* R_{\mathbf{T} \cap C_{\mathbf{G}}(s)^0 \cdot u}^{C_{\mathbf{G}}(s)^0 \cdot u} d_x f)(v)$$

which is the desired result if we apply Corollary 2.10 in $C_{\mathbf{G}}(s)^0 \cdot u$ and remark that by [DM94, 1.8 (iv)] the map $\mathbf{T}^1 \mapsto \mathbf{T} \cap C_{\mathbf{G}}(s)^0 \cdot u$ induces a bijection between $\{\mathbf{T}^1 \in \mathcal{T}_1^F \mid \mathbf{T} \ni s\}$ and F -stable “tori” of $C_{\mathbf{G}}(s)^0 \cdot u$. \square

Corollary 6.3. *A class function f on $(\mathbf{G}^1)^F$ is uniform if and only if for every $x \in (\mathbf{G}^1)^F$ the function $d_x f$ is uniform.*

Proof. Indeed, $f = p^{\mathbf{G}^1} f$ if and only if for any $x \in (\mathbf{G}^1)^F$ we have $d_x f = d_x p^{\mathbf{G}^1} f = p^{C_{\mathbf{G}}(s)^0 \cdot u} d_x f$, the last equality by Proposition 6.2. \square

For $x \in (\mathbf{G}^1)^F$ we consider the class function $\pi_x^{\mathbf{G}^1}$ on $(\mathbf{G}^1)^F$ defined by

$$\pi_x^{\mathbf{G}^1}(y) = \begin{cases} 0 & \text{if } y \text{ is not conjugate to } x \\ |C_{\mathbf{G}^0}(x)^F| & \text{if } y = x \end{cases}$$

Proposition 6.4. *For $x \in (\mathbf{G}^1)^F$ quasi-semi-simple the function $\pi_x^{\mathbf{G}^1}$ is uniform, given by*

$$\begin{aligned} \pi_x^{\mathbf{G}^1} &= \varepsilon_{\mathbf{G}^{x0}} |C_{\mathbf{G}}(x)^0|_p^{-1} \sum_{\{\mathbf{T}^1 \in \mathcal{T}_1^F \mid \mathbf{T}^1 \ni x\}} \varepsilon_{\mathbf{T}^1} R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}) \\ &= |W^0(x)|^{-1} \sum_{w \in W^0(x)} \dim R_{\mathbf{T}_w}^{C_{\mathbf{G}}(x)^0}(\text{Id}) R_{C_{\mathbf{G}^1}(\mathbf{T}_w)}^{\mathbf{G}^1}(\pi_x^{C_{\mathbf{G}^1}(\mathbf{T}_w)}) \end{aligned}$$

where in the second equality $W^0(x)$ denotes the Weyl group of $C_{\mathbf{G}}(x)^0$ and \mathbf{T}_w denotes an F -stable torus of type w of this last group.

Proof. First, using Corollary 6.3 we prove that $\pi_x^{\mathbf{G}^1}$ is uniform. Let su be the Jordan decomposition of x . For $y \in (\mathbf{G}^1)^F$ the function $d_y \pi_x^{\mathbf{G}^1}$ is zero unless the semi-simple part of y is conjugate to s . Hence it is sufficient to evaluate $d_y \pi_x^{\mathbf{G}^1}(v)$ for elements y whose semisimple part is equal to s . For such elements $d_y \pi_x^{\mathbf{G}^1}(v)$ is up to a coefficient equal to $\pi_u^{C_{\mathbf{G}}(s)^0 \cdot u}$. This function is uniform by [DM94, 4.14],

since u being the unipotent part of a quasi-semi-simple element is quasi-central in $C_{\mathbf{G}}(s)$ (see beginning of the proof of Proposition 5.1).

We have thus $\pi_x^{\mathbf{G}^1} = p^{\mathbf{G}^1} \pi_x^{\mathbf{G}^1}$. We use this to get the formula of the proposition. We start by using Proposition 2.13 to write $\pi_x^{\mathbf{G}^1} \text{St}_{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^1} \varepsilon_{\mathbf{G}^{x0}} |(\mathbf{G}^{x0})^F|_p \pi_x^{\mathbf{G}^1}$, or equivalently $\pi_x^{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^1} \varepsilon_{\mathbf{G}^{x0}} |(\mathbf{G}^{x0})^F|_p^{-1} p^{\mathbf{G}^1} (\pi_x^{\mathbf{G}^1} \text{St}_{\mathbf{G}^1})$. Using Corollary 2.10 and that by Proposition 5.1 we have $*R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1} \text{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1} \varepsilon_{\mathbf{T}^1} \text{St}_{\mathbf{T}^1} \text{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}(\pi_x^{\mathbf{G}^1})$, we get

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1} \text{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1} |(\mathbf{G}^1)^F|^{-1} \sum_{\mathbf{T}^1 \in \mathcal{T}_1^F} |(\mathbf{T}^1)^F| \varepsilon_{\mathbf{T}^1} R_{\mathbf{T}^1}^{\mathbf{G}^1}(\text{St}_{\mathbf{T}^1} \text{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}(\pi_x^{\mathbf{G}^1})).$$

The function $\text{St}_{\mathbf{T}^1}$ is constant equal to 1. Now we have

$$\text{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F} \pi_x^{\mathbf{G}^1} = |(\mathbf{T}^0)^F|^{-1} \sum_{\{g \in (\mathbf{G}^0)^F \mid gx \in \mathbf{T}^1\}} \pi_{gx}^{\mathbf{T}^1}.$$

To see this, do the scalar product with a class function f on $(\mathbf{T}^1)^F$:

$$\langle \text{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F} \pi_x^{\mathbf{G}^1}, f \rangle_{(\mathbf{T}^1)^F} = \langle \pi_x^{\mathbf{G}^1}, \text{Ind}_{\mathbf{T}^1}^{\mathbf{G}^1} f \rangle_{(\mathbf{G}^1)^F} = |(\mathbf{T}^0)^F|^{-1} \sum_{\{g \in (\mathbf{G}^0)^F \mid gx \in \mathbf{T}^1\}} f(gx).$$

We then get using that $|(\mathbf{T}^0)^F| = |(\mathbf{T}^1)^F|$

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1} \text{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1} |(\mathbf{G}^1)^F|^{-1} \sum_{\mathbf{T}^1 \in \mathcal{T}_1^F} \sum_{\{g \in (\mathbf{G}^0)^F \mid gx \in \mathbf{T}^1\}} \varepsilon_{\mathbf{T}^1} R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_{gx}^{\mathbf{T}^1}).$$

Taking $g^{-1}\mathbf{T}^1$ as summation index we get

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1} \text{St}_{\mathbf{G}^1}) = \varepsilon_{\mathbf{G}^1} \sum_{\{\mathbf{T}^1 \in \mathcal{T}_1^F \mid \mathbf{T}^1 \ni x\}} \varepsilon_{\mathbf{T}^1} R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}),$$

hence

$$\pi_x^{\mathbf{G}^1} = \varepsilon_{\mathbf{G}^{x0}} |(\mathbf{G}^{x0})^F|_p^{-1} \sum_{\{\mathbf{T}^1 \in \mathcal{T}_1^F \mid \mathbf{T}^1 \ni x\}} \varepsilon_{\mathbf{T}^1} R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}),$$

which is the first equality of the proposition.

For the second equality of the proposition, we first use [DM94, 1.8 (iii) and (iv)] to sum over tori of $C_{\mathbf{G}}(x)^0$: the $\mathbf{T}^1 \in \mathcal{T}_1^F$ containing x are in bijection with the maximal tori of $C_{\mathbf{G}}(x)^0$ by $\mathbf{T}^1 \mapsto (\mathbf{T}^1)^0$ and conversely $\mathbf{S} \mapsto C_{\mathbf{G}^1}(\mathbf{S})$. This bijection satisfies $\varepsilon_{\mathbf{T}^1} = \varepsilon_{\mathbf{S}}$ by definition of ε .

We then sum over $(C_{\mathbf{G}}(x)^0)^F$ -conjugacy classes of maximal tori, which are parameterized by F -conjugacy classes of $W^0(x)$. We then have to multiply by $|C_{\mathbf{G}}(x)^0|/|N_{(C_{\mathbf{G}}(x)^0)}(\mathbf{S})^F|$ the term indexed by the class of \mathbf{S} . Then we sum over the elements of $W^0(x)$. We then have to multiply the term indexed by w by $|C_{W^0(x)}(wF)|/|W^0(x)|$. Using $|N_{(C_{\mathbf{G}}(x)^0)}(\mathbf{S})^F| = |\mathbf{S}^F| |C_{W^0(x)}(wF)|$, and the formula for $\dim R_{\mathbf{T}_w}^{C_{\mathbf{G}}(x)^0}(\text{Id})$ we get the result. \square

7. CLASSIFICATION OF QUASI-SEMI-SIMPLE CLASSES

The first items of this section, before 7.7, apply for algebraic groups over an arbitrary algebraic closed field k .

We denote by $\mathcal{C}(\mathbf{G}^1)$ the set of conjugacy classes of \mathbf{G}^1 , that is the orbits under \mathbf{G}^0 -conjugacy, and denote by $\mathcal{C}(\mathbf{G}^1)_{\text{qss}}$ the set of quasi-semi-simple classes.

Proposition 7.1. *For $\mathbf{T}^1 \in \mathcal{T}_1$ write $\mathbf{T}^1 = \mathbf{T}^0 \cdot \sigma$ where σ is quasi-central. Then $\mathcal{C}(\mathbf{G}^1)_{\text{qss}}$ is in bijection with the set of $N_{\mathbf{G}^0}(\mathbf{T}^1)$ -orbits in \mathbf{T}^1 , which itself is in bijection with the set of W^σ -orbits in $\mathcal{C}(\mathbf{T}^1)$, where $W = N_{\mathbf{G}^0}(\mathbf{T}^0)/\mathbf{T}^0$. We have $\mathcal{C}(\mathbf{T}^1) \simeq \mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0)$ where \mathcal{L}_σ is the map $t \mapsto t^{-1} \cdot \sigma t$.*

Proof. By definition every quasi-semi-simple element of \mathbf{G}^1 is in some $\mathbf{T}^1 \in \mathcal{T}_1$ and \mathcal{T}_1 is a single orbit under \mathbf{G}^0 -conjugacy. It is thus sufficient to find how classes of \mathbf{G}^1 intersect \mathbf{T}^1 . By [DM94, 1.13] two elements of \mathbf{T}^1 are \mathbf{G}^0 -conjugate if and only if they are conjugate under $N_{\mathbf{G}^0}(\mathbf{T}^0)$. We can replace $N_{\mathbf{G}^0}(\mathbf{T}^0)$ by $N_{\mathbf{G}^0}(\mathbf{T}^1)$ since if $g(\sigma t) = \sigma t'$ where $g \in N_{\mathbf{G}^0}(\mathbf{T}^0)$ then the image of g in W lies in W^σ . By [DM94, 1.15(iii)] elements of W^σ have representatives in $\mathbf{G}^{\sigma 0}$. Write $g = s\dot{w}$ where \dot{w} is such a representative and $s \in \mathbf{T}^0$. Then ${}^{s\dot{w}}(t\sigma) = \mathcal{L}_\sigma(s^{-1})^w t\sigma$ whence the proposition. \square

Lemma 7.2. $\mathbf{T}^0 = \mathbf{T}^{\sigma 0} \cdot \mathcal{L}_\sigma(\mathbf{T}^0)$.

Proof. This is proved in [DM94, 1.33] when σ is unipotent (and then the product is direct). We proceed similarly to that proof: $\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)$ is finite, since its exponent divides the order of σ (if $\sigma(t^{-1}\sigma t) = t^{-1}\sigma t$ then $(t^{-1}\sigma t)^n = t^{-1}\sigma^n t$), and $\dim(\mathbf{T}^{\sigma 0}) + \dim(\mathcal{L}_\sigma(\mathbf{T}^0)) = \dim(\mathbf{T}^0)$ as the exact sequence $1 \rightarrow \mathbf{T}^{0\sigma} \rightarrow \mathbf{T}^0 \rightarrow \mathcal{L}_\sigma(\mathbf{T}^0) \rightarrow 1$ shows. \square

It follows that $\mathbf{T}^0/\mathcal{L}_\sigma(\mathbf{T}^0) \simeq \mathbf{T}^{\sigma 0}/(\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))$; since the set $\mathcal{C}(\mathbf{G}^{\sigma 0})_{\text{ss}}$ of semi-simple classes of $\mathbf{G}^{\sigma 0}$ identifies with the set of W^σ -orbits on $\mathbf{T}^{\sigma 0}$ this induces a surjective map $\mathcal{C}(\mathbf{G}^{\sigma 0})_{\text{ss}} \rightarrow \mathcal{C}(\mathbf{G}^1)_{\text{qss}}$.

Example 7.3. We will describe the quasi-semi-simple classes of $\mathbf{G}^0 \cdot \sigma$, where $\mathbf{G}^0 = \text{GL}_n(k)$ and σ is the quasi-central automorphism given by $\sigma(g) = J^t g^{-1} J^{-1}$, where,

if n is even J is the matrix $\begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$ with $J_0 = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$ and if n is odd

J is the antidiagonal matrix $\begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$ (any outer algebraic automorphism of

GL_n is equal to σ up to an inner automorphism).

The automorphism σ normalizes the pair $\mathbf{T}^0 \subset \mathbf{B}$ where \mathbf{T}^0 is the diagonal torus and \mathbf{B} the group of upper triangular matrices. Then $\mathbf{T}^1 = \mathbf{T}^0 \cdot \sigma \in \mathcal{T}_1$. For $\text{diag}(x_1, \dots, x_n) \in \mathbf{T}^0$, where $x_i \in k^\times$, we have $\sigma(\text{diag}(x_1, \dots, x_n)) = \text{diag}(x_n^{-1}, \dots, x_1^{-1})$. It follows that $\mathcal{L}_\sigma(\mathbf{T}^0) = \{\text{diag}(x_1, x_2, \dots, x_2, x_1)\}$ — here x_{m+1} is a square when $n = 2m+1$ but this is not a condition since k is algebraically closed. As suggested above, we could take as representatives of $\mathbf{T}^0/\mathcal{L}_\sigma(\mathbf{T}^0)$ the set $\mathbf{T}^{\sigma 0}/(\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))$, but since $\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)$ is not trivial (it consists of the diagonal matrices with entries ± 1 placed symmetrically), it is more convenient to take for representatives of the quasi-semi-simple classes the set $\{\text{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \dots, 1)\}\sigma$. In this model the action of W^σ is generated by the permutations of the $\lfloor \frac{n}{2} \rfloor$ first entries, and by the maps $x_i \mapsto x_i^{-1}$, so the quasi-semi-simple classes of $\mathbf{G}^0 \cdot \sigma$ are parameterized by the quasi-semi-simple classes of $\mathbf{G}^{\sigma 0}$.

Proposition 7.4. *Let $s\sigma = \text{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \dots, 1)\sigma$ be a quasi-semi-simple element as above. If $\text{char } k = 2$ then $C_{\mathbf{G}^0}(s\sigma)$ is connected. Otherwise, if n is odd, $A(s\sigma) := C_{\mathbf{G}^0}(s\sigma)/C_{\mathbf{G}^0}(s\sigma)^0$ is of order two, generated by $-1 \in Z\mathbf{G}^0 = Z\text{GL}_n(k)$.*

If n is even, $A(s\sigma) \neq 1$ if and only if for some i we have $x_i = -1$; then $x_i \mapsto x_i^{-1}$ is an element of W^σ which has a representative in $C_{\mathbf{G}^0}(s\sigma)$ generating $A(s\sigma)$, which is of order 2.

Proof. We will use that for a group G and an automorphism σ of G we have an exact sequence (see for example [St, 4.5])

$$(7.5) \quad 1 \rightarrow (ZG)^\sigma \rightarrow G^\sigma \rightarrow (G/ZG)^\sigma \rightarrow (\mathcal{L}_\sigma G \cap ZG)/\mathcal{L}_\sigma ZG \rightarrow 1$$

If we take $G = \mathbf{G}^0 = \mathrm{GL}_n(k)$ in 7.5 and $s\sigma$ for σ , since on $Z\mathbf{G}^0$ the map $\mathcal{L}_\sigma = \mathcal{L}_{s\sigma}$ is $z \mapsto z^2$, hence surjective, we get that $\mathbf{G}^{0s\sigma} \rightarrow \mathrm{PGL}_n^{s\sigma}$ is surjective and has kernel $(Z\mathbf{G}^0)^\sigma = \{\pm 1\}$.

Assume n odd and take $G = \mathrm{SL}_n(k)$ in 7.5. We have $Z\mathrm{SL}_n^\sigma = \{1\}$ so that we get the following diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{GL}_n^{s\sigma} & \longrightarrow & \mathrm{PGL}_n^{s\sigma} \longrightarrow 1 \\ & & & & \uparrow & & \parallel \\ & & & & \mathrm{SL}_n^{s\sigma} & \longrightarrow & \mathrm{PGL}_n^{s\sigma} \longrightarrow 1 \end{array}$$

This shows that $\mathrm{GL}_n^{s\sigma}/\mathrm{SL}_n^{s\sigma} \simeq \{\pm 1\}$; by [St, 8.1] $\mathrm{SL}_n^{s\sigma}$ is connected, hence $\mathrm{PGL}_n^{s\sigma}$ is connected thus $\mathrm{GL}_n^{s\sigma} = (\mathrm{GL}_n^{s\sigma})^0 \times \{\pm 1\}$ is connected if and only if $\mathrm{char} k = 2$.

Assume now n even; then $(\mathbf{T}^0)^\sigma$ is connected hence $-1 \in (\mathrm{GL}_n^{s\sigma})^0$ for all $s \in \mathbf{T}_0$. Using this, the exact sequence $1 \rightarrow \{\pm 1\} \rightarrow \mathrm{GL}_n^{s\sigma} \rightarrow \mathrm{PGL}_n^{s\sigma} \rightarrow 1$ implies $A(s\sigma) = \mathbf{G}^{s\sigma}/\mathbf{G}^{0s\sigma} = \mathrm{GL}_n^{s\sigma}/(\mathrm{GL}_n^{s\sigma})^0 \simeq \mathrm{PGL}_n^{s\sigma}/(\mathrm{PGL}_n^{s\sigma})^0$. To compute this group we use 7.5 with $\mathrm{SL}_n(k)$ for G and $s\sigma$ for σ :

$$1 \rightarrow \{\pm 1\} \rightarrow \mathrm{SL}_n^{s\sigma} \rightarrow \mathrm{PGL}_n^{s\sigma} \rightarrow (\mathcal{L}_{s\sigma} \mathrm{SL}_n \cap Z \mathrm{SL}_n)/\mathcal{L}_\sigma Z \mathrm{SL}_n \rightarrow 1$$

which, since $\mathrm{SL}_n^{s\sigma}$ is connected, implies that $A(s\sigma) = (\mathcal{L}_{s\sigma} \mathrm{SL}_n \cap Z \mathrm{SL}_n)/\mathcal{L}_\sigma Z \mathrm{SL}_n$ thus is non trivial (of order 2) if and only if $\mathcal{L}_{s\sigma} \mathrm{SL}_n \cap Z \mathrm{SL}_n$ contains an element which is not a square in $Z \mathrm{SL}_n$; thus $A(s\sigma)$ is trivial if $\mathrm{char} k = 2$. We assume now $\mathrm{char} k \neq 2$. Then a non-square is of the form $\mathrm{diag}(z, \dots, z)$ with $z^m = -1$ if we set $m = n/2$.

The following lemma is a transcription of [St, 9.5].

Lemma 7.6. *Let σ be a quasi-central automorphism of the connected reductive group \mathbf{G} which stabilizes the pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup; let W be the Weyl group of \mathbf{T} and let $s \in \mathbf{T}$. Then $\mathbf{T} \cap \mathcal{L}_{s\sigma}(\mathbf{G}) = \{\mathcal{L}_w(s^{-1}) \mid w \in W^\sigma\} \cdot \mathcal{L}_\sigma(\mathbf{T})$.*

Proof. Assume $t = \mathcal{L}_{s\sigma}(x)$ for $t \in \mathbf{T}$, or equivalently $xt = {}^{s\sigma}x$. Then if x is in the Bruhat cell $\mathbf{B}w\mathbf{B}$, we must have $w \in W^\sigma$. Taking for w a σ -stable representative \dot{w} and writing the unique Bruhat decomposition $x = u_1 \dot{w} t_1 u_2$ where $u_2 \in \mathbf{U}$, $t_1 \in \mathbf{T}$ and $u_1 \in \mathbf{U} \cap {}^w \mathbf{U}^-$ where \mathbf{U} is the unipotent radical of \mathbf{B} and \mathbf{U}^- the unipotent radical of the opposite Borel, the equality $xt = {}^{s\sigma}x$ implies that $\dot{w} t_1 t = {}^{s\sigma}(\dot{w} t_1)$ or equivalently $t = \mathcal{L}_{w^{-1}}(s^{-1}) \mathcal{L}_\sigma(t_1)$, whence the lemma. \square

From this lemma we get $\mathcal{L}_{s\sigma}(\mathrm{SL}_n) \cap Z \mathrm{SL}_n = \{\mathcal{L}_w(s^{-1}) \mid w \in W^\sigma\} \cdot \mathcal{L}_\sigma(\mathbf{T}) \cap Z \mathrm{SL}_n$, where \mathbf{T} is the maximal torus of SL_n . The element $\mathrm{diag}(x_1, x_2, \dots, x_m, 1, \dots, 1)\sigma$ is conjugate to $s\sigma = \mathrm{diag}(y_1, y_2, \dots, y_m, y_m^{-1}, \dots, y_1^{-1})\sigma \in \mathbf{T}^\sigma \cdot \sigma$ where $y_i^2 = x_i$. It will have a non connected centralizer if and only if for some $w \in W^\sigma$ and some $t \in \mathbf{T}$ we have $\mathcal{L}_w(s^{-1}) \cdot \mathcal{L}_\sigma(t) = \mathrm{diag}(z, \dots, z)$ with $z^m = -1$ and then an appropriate representative of w (multiplying if needed by an element of $Z \mathrm{GL}_n$) will be

in $C_{\mathbf{G}^0}(s\sigma)$ and have a non-trivial image in $A(s\sigma)$. Since s and w are σ -fixed, we have $\mathcal{L}_w(s) \in \mathbf{T}^\sigma$, thus is of the form $\text{diag}(a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1})$. Since $\mathcal{L}_\sigma(\mathbf{T}) = \{\text{diag}(t_1, \dots, t_m, t_m, \dots, t_1) \mid t_1 t_2 \dots t_m = 1\}$, we get $z = a_1 t_1 = a_2 t_2 = \dots = a_m t_m = a_m^{-1} t_m = \dots = a_1^{-1} t_1$; in particular $a_i = \pm 1$ for all i and $a_1 a_2 \dots a_m = -1$. We can take w up to conjugacy in W^σ since $\mathcal{L}_{v w v^{-1}}(s^{-1}) = {}^v \mathcal{L}_w(v^{-1} s^{-1})$ and $\mathcal{L}_\sigma(\mathbf{T})$ is invariant under W^σ -conjugacy. We see W^σ as the group of permutations of $\{1, 2, \dots, m, -m, \dots, -1\}$ which preserves the pairs $\{i, -i\}$. A non-trivial cycle of w has, up to conjugacy, the form either $(1, -1)$ or $(1, -2, 3, \dots, (-1)^{i-1} i, i+1, i+2, \dots, k)$ with $0 \leq i \leq k \leq n$ and i even, or $(1, -2, 3, \dots, (-1)^{i-1} i, -(i+1), -(i+2), \dots, -k, -1, 2, -3, \dots, k)$ and i odd. The contribution to $a_1 \dots a_m$ of the orbit $(1, -1)$ is $a_1 = y_1^2$ hence is 1 except if $y_1^2 = x_1 = -1$. Let us consider an orbit of the second form. The k first coordinates of $\mathcal{L}_w(s^{-1})$ are $(y_1 y_2, \dots, y_i y_{i+1}, y_{i+1}/y_{i+2}, \dots, y_k/y_1)$. Hence there must exist signs ε_j such that $y_2 = \varepsilon_1/y_1$, $y_3 = \varepsilon_2/y_2, \dots$, $y_{i+1} = \varepsilon_i/y_i$ and $y_{i+2} = \varepsilon_{i+1} y_{i+1}, \dots$, $y_k = \varepsilon_{k-1} y_{k-1}$, $y_1 = \varepsilon_k y_k$. This gives $y_1 = \begin{cases} \varepsilon_1 \dots \varepsilon_k y_1 & \text{if } i \text{ is even} \\ \varepsilon_1 \dots \varepsilon_k / y_1 & \text{if } i \text{ is odd} \end{cases}$. The contribution of the orbit to $a_1 \dots a_m$ is $\varepsilon_1 \dots \varepsilon_k$ thus is 1 if i is even and $x_1 = y_1^2$ if i is odd. Again, we see that one of the x_i must equal -1 to get $a_1 \dots a_m = -1$. Conversely if $x_1 = -1$, for any z such that $z^m = -1$, choosing t such that $\mathcal{L}_\sigma(t) = \text{diag}(-z, z, z, \dots, z, -z)$ and taking $w = (1, -1)$ we get $\mathcal{L}_w(s^{-1})\mathcal{L}_\sigma(t) = \text{diag}(z, \dots, z)$ as desired. \square

We now go back to the case where $k = \overline{\mathbb{F}}_q$, and in the context of Proposition 7.1, we now assume that \mathbf{T}^1 is F -stable and that σ induces an F -stable automorphism of \mathbf{G}^0 .

Proposition 7.7. *Let $\mathbf{T}^{1rat} = \{s \in \mathbf{T}^1 \mid \exists n \in N_{\mathbf{G}^0}(\mathbf{T}^1), {}^n F s = s\}$; then \mathbf{T}^{1rat} is stable by \mathbf{T}^0 -conjugacy, which gives a meaning to $\mathcal{C}(\mathbf{T}^{1rat})$. Then $c \mapsto c \cap \mathbf{T}^1$ induces a bijection between $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$ and the W^σ -orbits on $\mathcal{C}(\mathbf{T}^{1rat})$.*

Proof. A class $c \in \mathcal{C}(\mathbf{G}^1)_{qss}$ is F -stable if and only if given $s \in c$ we have ${}^F s \in c$. If we take $s \in c \cap \mathbf{T}^1$ then ${}^F s \in c \cap \mathbf{T}^1$ which as observed in the proof of 7.1 implies that ${}^F s$ is conjugate to s under $N_{\mathbf{G}^0}(\mathbf{T}^1)$, that is $s \in \mathbf{T}^{1rat}$. Thus c is F -stable if and only if $c \cap \mathbf{T}^1 = c \cap \mathbf{T}^{1rat}$. The proposition then results from Proposition 7.1 observing that \mathbf{T}^{1rat} is stable under $N_{\mathbf{G}^0}(\mathbf{T}^1)$ -conjugacy and that the corresponding orbits are the W^σ -orbits on $\mathcal{C}(\mathbf{T}^{1rat})$. \square

Example 7.8. When $\mathbf{G}^1 = \text{GL}_n(\overline{\mathbb{F}}_q) \cdot \sigma$ with σ as in Example 7.3, the map

$$\text{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \dots, 1) \mapsto \text{diag}(x_1, x_2, \dots, x_{\lfloor \frac{n}{2} \rfloor}, \dagger, x_{\lfloor \frac{n}{2} \rfloor}^{-1}, \dots, x_2^{-1}, x_1^{-1})$$

where \dagger represents 1 if n is odd and an omitted entry otherwise, is compatible with the action of W^σ as described in 7.3 on the left-hand side and the natural action on the right-hand side. This map induces a bijection from $\mathcal{C}(\mathbf{G}^1)_{qss}$ to the semi-simple classes of $(\text{GL}_n^\sigma)^0$ which restricts to a bijection from $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$ to the F -stable semi-simple classes of $(\text{GL}_n^\sigma)^0$.

We will now compute the cardinality of $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$.

Proposition 7.9. *Let f be a function on $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$. Then*

$$\sum_{c \in (\mathcal{C}(\mathbf{G}^1)_{qss})^F} f(c) = |W^\sigma|^{-1} \sum_{w \in W^\sigma} \tilde{f}(w)$$

where $\tilde{f}(w) := \sum_s f(s)$, where s runs over representatives in \mathbf{T}^{1wF} of $\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}$.

Proof. We have $\mathcal{C}(\mathbf{T}^{1\text{rat}}) = \bigcup_{w \in W^\sigma} \{s\mathcal{L}_\sigma(\mathbf{T}^0) \in \mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0) \mid s\mathcal{L}_\sigma(\mathbf{T}^0) \text{ is } wF\text{-stable}\}$. The conjugation by $v \in W^\sigma$ sends a wF -stable coset $s\mathcal{L}_\sigma(\mathbf{T}^0)$ to a $vwFv^{-1}$ -stable coset; and the number of w such that $s\mathcal{L}_\sigma(\mathbf{T}^0)$ is wF -stable is equal to $N_{W^\sigma}(s\mathcal{L}_\sigma(\mathbf{T}^0))$. It follows that

$$\sum_{c \in (\mathcal{C}(\mathbf{G}^1)_{\text{qss}})^F} f(c) = |W^\sigma|^{-1} \sum_{w \in W^\sigma} \sum_{s\mathcal{L}_\sigma(\mathbf{T}^0) \in (\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}} f(s\mathcal{L}_\sigma(\mathbf{T}^0)).$$

The proposition follows since, $\mathcal{L}_\sigma(\mathbf{T}^0)$ being connected, we have $(\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0))^{wF} = \mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}$. \square

Corollary 7.10. *We have $|(\mathcal{C}(\mathbf{G}^1)_{\text{qss}})^F| = |(\mathcal{C}(\mathbf{G}^{\sigma^0})_{\text{ss}})^F|$.*

Proof. Let us take $f = 1$ in 7.9. We need to sum over $w \in W^\sigma$ the value $|\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|$. First note that $|\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}| = |\mathbf{T}^{0wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|$. By Lemma 7.2 we have the exact sequence

$$1 \rightarrow \mathbf{T}^{\sigma^0} \cap \mathcal{L}_\sigma(\mathbf{T}^0) \rightarrow \mathbf{T}^{\sigma^0} \times \mathcal{L}_\sigma(\mathbf{T}^0) \rightarrow \mathbf{T}^0 \rightarrow 1$$

whence the Galois cohomology exact sequence:

$$1 \rightarrow (\mathbf{T}^{\sigma^0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))^{wF} \rightarrow \mathbf{T}^{\sigma^0 wF} \times (\mathcal{L}_\sigma(\mathbf{T}^0))^{wF} \rightarrow \mathbf{T}^{0wF} \rightarrow H^1(wF, (\mathbf{T}^{\sigma^0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))) \rightarrow 1.$$

Using that for any automorphism τ of a finite group G we have $|G^\tau| = |H^1(\tau, G)|$, we have $|(\mathbf{T}^{\sigma^0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))^{wF}| = |H^1(wF, (\mathbf{T}^{\sigma^0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)))|$. Together with the above exact sequence it implies that $|\mathbf{T}^{0wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}| = |\mathbf{T}^{\sigma^0 wF}|$ whence

$$|(\mathcal{C}(\mathbf{G}^1)_{\text{qss}})^F| = |W^\sigma|^{-1} \sum_{w \in W^\sigma} |\mathbf{T}^{\sigma^0 wF}|.$$

The corollary follows by either applying the same formula for the connected group \mathbf{G}^{σ^0} , or referring to [Le, Proposition 2.1]. \square

8. SHINTANI DESCENT.

We now look at Shintani descent in our context; we are able to obtain a result when $\mathbf{G}^1/\mathbf{G}^0$ is semi-simple. We should mention previous work on this subject: Eftekhari ([E96, II. 3.4]) has the same result in the case of a torus; he does not need to assume p good but needs q to be large enough to apply results of Lusztig identifying Deligne-Lusztig induction with induction of character sheaves; Digne ([D99, 1.1]) has the result in the same generality excepted that he needs the assumption that \mathbf{G}^1 contains an F -stable quasi-central element; however a defect of his proof is the use without proof of the property given in Lemma 8.4 below.

As above \mathbf{G}^1 denotes an F -stable connected component of \mathbf{G} of the form $\mathbf{G}^0 \cdot \sigma$ where σ induces a quasi-central automorphism of \mathbf{G}^0 commuting with F .

Applying Lang's theorem, one can write any element of \mathbf{G}^1 as $x \cdot {}^\sigma F x^{-1} \sigma$ for some $x \in \mathbf{G}^0$, or as $\sigma {}^F x^{-1} \cdot x$ for some $x \in \mathbf{G}^0$. Using that σ , as automorphism, commutes with F , it is easy to check that the correspondence $x \cdot {}^\sigma F x^{-1} \sigma \mapsto \sigma {}^F x^{-1} \cdot x$ induces a bijection $n_{F/\sigma F}$ from the $(\mathbf{G}^0)^F$ -conjugacy classes of $(\mathbf{G}^1)^F$ to the $\mathbf{G}^{0\sigma F}$ -conjugacy classes of $(\mathbf{G}^1)^{\sigma F}$ and that $|\mathbf{G}^{0\sigma F}||c| = |(\mathbf{G}^0)^F||n_{F/\sigma F}(c)|$ for any $(\mathbf{G}^0)^F$ -class c in $(\mathbf{G}^1)^F$. It follows that the operator $\text{sh}_{F/\sigma F}$ from $(\mathbf{G}^0)^F$ -class functions on $(\mathbf{G}^1)^F$

to $\mathbf{G}^{0\sigma F}$ -class functions on $(\mathbf{G}^1)^{\sigma F}$ defined by $\mathrm{sh}_{F/\sigma F}(\chi)(n_{F/\sigma F}x) = \chi(x)$ is an isometry.

The end of this section is devoted to the proof of the following

Proposition 8.1. *Let $\mathbf{L}^1 = N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ be a “Levi” of \mathbf{G}^1 containing σ , where \mathbf{L}^0 is F -stable; we have $\mathbf{L}^1 = \mathbf{L}^0 \cdot \sigma$. Assume that σ is semi-simple and that the characteristic is good for $\mathbf{G}^{\sigma 0}$. Then*

$$\mathrm{sh}_{F/\sigma F} \circ {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} = {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ \mathrm{sh}_{F/\sigma F} \quad \text{and} \quad \mathrm{sh}_{F/\sigma F} \circ R_{\mathbf{L}^1}^{\mathbf{G}^1} = R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ \mathrm{sh}_{F/\sigma F}.$$

Proof. The second equality follows from the first by adjunction, using that the adjoint of $\mathrm{sh}_{F/\sigma F}$ is $\mathrm{sh}_{F/\sigma F}^{-1}$. Let us prove the first equality.

Let χ be a $(\mathbf{G}^0)^F$ -class function on \mathbf{G}^1 and let $\sigma l u = u \sigma l$ be the Jordan decomposition of an element of $(\mathbf{L}^1)^{\sigma F}$ with u unipotent and σl semi-simple. By the character formula 2.5(iii) and the definition of $Q_{\mathbf{L}^0}^{\sigma 0}$ we have

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = |(\mathbf{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v) \mathrm{Trace}((v, u^{-1}) | H_c^*(Y_{\mathbf{U}, \sigma F})),$$

where v (resp. u) acts by left- (resp. right-) translation on $Y_{\mathbf{U}, \sigma F} = \{x \in (\mathbf{G}^{\sigma l})^0 \mid x^{-1} \cdot {}^{\sigma F}x \in \mathbf{U}\}$ where \mathbf{U} denotes the unipotent radical of \mathbf{P}^0 ; in the summation v is in the identity component of $\mathbf{G}^{\sigma l}$ since, σ being semi-simple, u is in \mathbf{G}^0 hence in $(\mathbf{G}^{\sigma l})^0$ by [DM94, 1.8 (i)] since σl is semisimple.

Let us write $l = {}^F\lambda^{-1} \cdot \lambda$ with $\lambda \in \mathbf{L}^0$, so that $\sigma l = n_{F/\sigma F}(l'\sigma)$ where $l' = \lambda \cdot {}^{\sigma F}\lambda^{-1}$.

Lemma 8.2. *For $v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}$ we have $\sigma l v = n_{F/\sigma F}((\sigma l \cdot v')^{\sigma F\lambda^{-1}})$ where $v' = n_{\sigma F/\sigma F}v \in (\mathbf{G}^{\sigma l})^{0\sigma F}$ is defined by writing $v = {}^{\sigma F}\eta \cdot \eta^{-1}$ where $\eta \in (\mathbf{G}^{\sigma l})^0$ and setting $v' = \eta^{-1} \cdot {}^{\sigma F}\eta$.*

Proof. We have $\sigma l v = \sigma l {}^{\sigma F}\eta \cdot \eta^{-1} = {}^{\sigma F}\eta \sigma l \eta^{-1} = {}^{\sigma F}(\eta \lambda^{-1}) \lambda \eta^{-1}$, thus $\sigma l v = n_{F/\sigma F}((\lambda \eta^{-1}) \cdot {}^{\sigma F}(\eta \lambda^{-1}) \sigma)$. And we have $(\lambda \eta^{-1}) \cdot {}^{\sigma F}(\eta \lambda^{-1}) \sigma = \lambda v' {}^{\sigma F}\lambda^{-1} \sigma = {}^F\lambda v' {}^{\sigma F}\lambda^{-1} = (\sigma l v')^{\sigma F\lambda^{-1}}$, thus $\mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v) = \chi((\sigma l v')^{\sigma F\lambda^{-1}})$. \square

Lemma 8.3. (i) *We have $(\sigma l)^{\sigma F\lambda^{-1}} = l'\sigma$.*

(ii) *The conjugation $x \mapsto x^{\sigma F\lambda^{-1}}$ maps $\mathbf{G}^{\sigma l}$ and the action of σF on it, to $\mathbf{G}^{l'\sigma}$ with the action of F on it; in particular it induces bijections $(\mathbf{G}^{\sigma l})^{0\sigma F} \xrightarrow{\sim} (\mathbf{G}^{l'\sigma})^{0F}$ and $Y_{\mathbf{U}, \sigma F} \xrightarrow{\sim} Y_{\mathbf{U}, F}$, where $Y_{\mathbf{U}, F} = \{x \in (\mathbf{G}^{l'\sigma})^0 \mid x^{-1}Fx \in \mathbf{U}\}$.*

Proof. (i) is an obvious computation and shows that if $x \in \mathbf{G}^{\sigma l}$ then $x^{\sigma F\lambda^{-1}} \in \mathbf{G}^{l'\sigma}$. To prove (ii), it remains to show that if $x \in \mathbf{G}^{\sigma l}$ then ${}^F(x^{\sigma F\lambda^{-1}}) = (\sigma F x)^{\sigma F\lambda^{-1}}$. From $x^{\sigma} = x^{l^{-1}} = x^{\lambda^{-1} \cdot {}^F\lambda}$, we get $x^{\sigma F\lambda^{-1}} = x^{\lambda^{-1}}$, whence ${}^F(x^{\sigma F\lambda^{-1}}) = ({}^F x)^{F\lambda^{-1}} = ((\sigma F x)^{\sigma})^{F\lambda^{-1}} = (\sigma F x)^{\sigma F\lambda^{-1}}$. \square

Applying lemmas 8.2 and 8.3 we get

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = |(\mathbf{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \chi((\sigma l v')^{\sigma F\lambda^{-1}}) \mathrm{Trace}((v^{\sigma F\lambda^{-1}}, (u^{\sigma F\lambda^{-1}})^{-1}) | H_c^*(Y_{\mathbf{U}, F})).$$

Lemma 8.4. *Assume that the characteristic is good for \mathbf{G}^{σ^0} , where σ is a quasi-central element of \mathbf{G} . Then it is also good for $(\mathbf{G}^s)^0$ where s is any quasi-semi-simple element of $\mathbf{G}^0 \cdot \sigma$.*

Proof. Let Σ_σ (resp. Σ_s) be the root system of \mathbf{G}^{σ^0} (resp. $(\mathbf{G}^s)^0$). By definition, a characteristic p is good for a reductive group if for no closed subsystem of its root system the quotient of the generated lattices has p -torsion. The system Σ_s is not a closed subsystem of Σ_σ in general, but the relationship is expounded in [DM02]: let Σ be the root system of \mathbf{G}^0 with respect to a σ -stable pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup of \mathbf{G}^0 . Up to conjugacy, we may assume that s also stabilizes that pair. Let $\overline{\Sigma}$ the set of sums of the σ -orbits in Σ , and Σ' the set of averages of the same orbits. Then Σ' is a non-necessarily reduced root system, but Σ_σ and Σ_s are subsystems of Σ' and are reduced. The system $\overline{\Sigma}$ is reduced, and the set of sums of orbits whose average is in Σ_σ (resp. Σ_s) is a closed subsystem that we denote by $\overline{\Sigma}_\sigma$ (resp. $\overline{\Sigma}_s$).

We need now the following generalization of [Bou, chap VI, §1.1, lemme1]

Lemma 8.5. *Let \mathcal{L} be a finite set of lines generating a vector space V over a field of characteristic 0; then two reflections of V which stabilize \mathcal{L} and have a common eigenvalue $\zeta \neq 1$ with ζ -eigenspace the same line of \mathcal{L} are equal.*

Proof. Here we mean by reflection an element $s \in \mathrm{GL}(V)$ such that $\ker(s - 1)$ is a hyperplane. Let s and s' be reflections as in the statement. The product $s^{-1}s'$ stabilizes \mathcal{L} , so has a power which fixes \mathcal{L} , thus is semisimple. On the other hand $s^{-1}s'$ by assumption fixes one line $L \in \mathcal{L}$ and induces the identity on V/L , thus is unipotent. Being semi-simple and unipotent it has to be the identity. \square

It follows from 8.5 that two root systems with proportional roots have same Weyl group, thus same good primes; thus:

- Σ_s and $\overline{\Sigma}_s$ have same good primes, as well as Σ_σ and $\overline{\Sigma}_\sigma$.
- The bad primes for $\overline{\Sigma}_s$ are a subset of those for $\overline{\Sigma}$, since it is a closed subsystem.

It only remains to show that the good primes for $\overline{\Sigma}$ are the same as for $\overline{\Sigma}_\sigma$, which can be checked case by case: we can reduce to the case where Σ is irreducible, where these systems coincide excepted when Σ is of type A_{2n} ; but in this case $\overline{\Sigma}$ is of type B_n and Σ_σ is of type B_n or C_n , which have the same set $\{2\}$ of bad primes. \square

Since the characteristic is good for \mathbf{G}^{σ^0} , hence also for $(\mathbf{G}^{\sigma^l})^0$ by lemma 8.4, the elements v' et v are conjugate in $(\mathbf{G}^{\sigma^l})^{0\sigma^F}$ (see [DM85, IV Corollaire 1.2]). By Lemma 8.3(ii), the element $v^{\sigma^F\lambda^{-1}}$ runs over the unipotent elements of $(\mathbf{G}^{l'\sigma})^{0^F}$ when v runs over $(\mathbf{G}^{\sigma^l})^{0\sigma^F}_{\mathrm{unip}}$. Using moreover the equality $|(\mathbf{G}^{\sigma^l})^{0\sigma^F}| = |(\mathbf{G}^{l'\sigma})^{0^F}|$ we get

$$(*) \quad (*R_{\mathbf{L}^1}^{\mathbf{G}^1} \mathrm{sh}_{F/\sigma^F}(\chi))(\sigma l u) = \frac{1}{|(\mathbf{G}^{l'\sigma})^{0^F}|} \sum_{u_1 \in (\mathbf{G}^{l'\sigma})^{0^F}_{\mathrm{unip}}} \chi(u_1 l' \sigma) \mathrm{Trace}((u_1, (u^{\sigma^F\lambda^{-1}})^{-1}) | H_c^*(Y_{\mathbf{U}, F}))$$

On the other hand by Lemma 8.2 applied with $v = u$, we have

$$(\mathrm{sh}_{F/\sigma^F} * R_{\mathbf{L}^1}^{\mathbf{G}^1}(\chi))(\sigma l u) = *R_{\mathbf{L}^1}^{\mathbf{G}^1}(\chi)((\sigma l u)^{\sigma^F\lambda^{-1}}) = *R_{\mathbf{L}^1}^{\mathbf{G}^1}(\chi)(l' \sigma \cdot u^{\sigma^F\lambda^{-1}}),$$

the second equality by Lemma 8.3(i). By the character formula this is equal to the right-hand side of formula (*). \square

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